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Aberration Diffraction Effects

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IV.—*Aberration Diffraction Effects.*

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(*Communicated by Prof.* A. S. Eddington, *F.R.S.*)

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Introductory.

The image of a luminous point, given by a symmetrical optical system, will not itself be a point; and this will follow both from the nature of light and also from the necessary 'imperfections' of the system. A part only of the incident wave will pass through the system and diffraction phenomena will appear; in addition the emergent wave will not be a portion of a sphere but will be distorted by the geometrical aberrations of the system. Diffraction theory would indicate that corresponding to a point source of light a system of luminous rings should be produced upon the image plane; this was investigated by Airy in 1834; geometrical theory, on the other hand, leads to a consideration of several types and orders of aberration, the more common ones being better known as the 'Five Aberrations of Von Seidel.' They are:—Spherical Aberration, Coma, Astigmatism, Curvature of the Field and Distortion; these are well known and they have been investigated by a number of writers. In the present paper a consideration is undertaken of the modification of the 'ideal' diffraction pattern produced by these geometrical aberrations. The method adopted depends upon the Eikonal Function of Bruns, and a summary of the properties of this function is given, therefore, in Part I of the paper; Part II deals with the Aberration-Diffraction effects.

Throughout Parts I and II of the paper it is assumed that the stops of the optical system are circular, with centres upon the axis of symmetry; and this is generally the case. Occasionally, however, other stops are used, and in Part III of the paper is undertaken a consideration of the diffraction effects of such. The precise forms of aperture considered are the following:—

1. the usual circular aperture with the central portion stopped out—as suggested by Lord Rayleigh:
2. one (or two parallel) narrow rectangular apertures—as used for the determination of the diameters of large stars and for the separation of close double-stars:
3. a semi-circular aperture—as used in a heliometer.

The detailed examination of a certain general function which arises in the course of the investigation is given in an Appendix and the results obtained are quoted in the text as required.

References.

Part I.—The form of the Eikonal Function is due, in the main, to Mr. T. Smith, of the National Physical Laboratory, although the presentation is my own. Reference may be made to the following :—

"The Changes in the Aberrations when the Object and Stop are moved": T. Smith, 'Trans. Optical Society,' Vol. XXIII, 5, 1921–22.

Art. "Optical Calculations": 'Dictionary of Applied Physics.'

Part II.—Rayleigh : 'Coll. Wks.,' I, II. Conrady, A. E.: "Star Discs," 'M.N.R.A.S.,' June, 1919. Buxton, A.: "Star Discs," 'M.N.R.A.S.,' June, 1921. Martin, L. C.: "Star Discs," 'M.N.R.A.S.,' March, 1922.

Part III (Non-Circular aperture).—Rayleigh : 'Coll. Wks.,' III, p. 90.

I regret that, although this paper was read before the Society in March, 1923, I have been unable to prepare it for publication until the present month (June, 1925).

Part I.

§ 1. Let there be n coaxial surfaces of revolution and let μ_λ be the refractive index of the medium between the surfaces λ, $\lambda + 1$; let κ_λ, $\kappa_{1\lambda}$ be the powers of the surface λ, and of the surfaces 1 to λ respectively ; let F, F′ be the principal foci of the system and x, x' distances measured from F, F′ respectively to a pair of conjugate points. Then

$$xx' = -\frac{\mu_0\mu_\lambda}{\kappa^2_{1,n}}, \quad \ldots \quad (1)$$

where x, x' are measured positively in the direction of the incident light. Again, let $J_\lambda \equiv \frac{\kappa_\lambda}{\mu_{\lambda-1}\mu_\lambda}$, $J_{1,\lambda} \equiv \frac{\kappa_{1,\lambda}}{\mu_0\mu_\lambda}$, so that the symbol J denotes modified power. If J ($\equiv J_{1,n}$) be the modified power of the whole system, and $X = \mu_0 x J$, $X' = \mu_n x' J$, then (1) may be written

$$XX' = -1. \quad \ldots \quad (2)$$

If m be the modified magnification, *i.e.*, the ratio of the reduced dimensions of image and object, then

$$X = \frac{1}{m}, \quad X' = -m. \quad \ldots \quad (3)$$

A notation has now been obtained in which the continued recurrence of the quantities κ, μ is avoided ; and it is to be noticed that X, X′ are pure numbers. If two systems of powers J_1, J_2, with principal foci at F_1, F_1' and F_2, F_2', respectively form one system of power J and principal foci F, F′, then

$$F_1'F_2 = -\frac{J}{J_1J_2}, \quad F'F_2' = +\frac{J_1}{J_2J}, \quad F_1F = +\frac{J_2}{JJ_1}.$$

§ 2. *The Eikonal of Bruns.*—Let O, O′, taken as origins of co-ordinates, be in the object-space and image space respectively and upon the axis of a symmetrical optical system; let this axis be taken as axis of x, x' in the two spaces. Let the remaining orthogonal axes be parallel. Let a ray whose direction cosines are (L, M, N), (L′, M′, N′) respectively in the two spaces cut the planes (yz) ($y'z'$) in the points (y, z) (y', z'); let P, P′ be the feet of the perpendiculars from O, O′ upon the two portions of the ray. Then the reduced distance PP′, measured along the ray, is defined to be the 'Eikonal' E and it follows that

$$y = \frac{\partial E}{\partial M}, \qquad z = \frac{\partial E}{\partial N}, \quad y' = -\frac{\partial E}{\partial M'}, \qquad z' = -\frac{\partial E}{\partial N'}.$$

Let e be the reduced distance OO′, measured along the axis, and write

$$J\,(E - e) = U\,;$$

then

$$Y = \frac{\partial U}{\partial M}, \qquad Z = \frac{\partial U}{\partial N}, \quad Y' = -\frac{\partial U}{\partial M'}, \qquad Z' = -\frac{\partial U}{\partial N'}, \qquad . \quad . \quad . \quad (1)$$

and U is known as the 'modified eikonal' or the 'eikonal expressed as a ratio'; it represents the difference between the reduced paths PP′ and OO′. U is a function of four variables which may conveniently be M, N, M′, N′; but because of the symmetry of the system the three variables (1), (2), (3) are sufficient, given by

$$M^2 + N^2 = (1), \qquad MM' + NN' = (2), \qquad M'^2 + N'^2 = (3).$$

If the principal foci be taken as base points (O, O′) U is known as the 'focal eikonal'; it is given by

$$U = -(2) + \text{terms of higher orders},$$

and the focal eikonal for a single spherical surface is given by

$$\begin{aligned} U = -(2) - &\tfrac{1}{8}\,[v\{(1) - 2\,(2) + (3)\}^2 + 2\,(1)\,(3)] \\ &+ \tfrac{1}{16}\,[v^2\{(1) - 2\,(2) + (3)\}^2 \\ &\qquad - v\{(1) - 2\,(2) + (3)\}\,\{(1)^2 - 2\,(1)\,(3) + (3)^2\} - (1)^2(3) - (1)(2)^2] \\ &+ \text{terms of higher orders,} \end{aligned}$$

where $v = \dfrac{\mu\mu'}{(\mu' - \mu)^2}$ and the surface separates media of indices μ, μ'.

§ 3. Let now O, O′ be conjugate points associated with modified magnification m, and let s be the modified magnification for the axial points of the entrance and exit pupils of the system: let α, β, γ be new variables defined as follows:—

$$\alpha\,(s - m)^2 = (M - sM')^2 + (N - sN')^2 = (1) - 2s\,(2) + s^2\,(3)$$

$$\beta\,(s - m)^2 = (M - sM')\,(M - mM') + (N - sN')\,(N - mN') = (1) - (s + m)\,(2) + sm\,(3)$$

$$\gamma\,(s - m)^2 = (M - mM')^2 + (N - mN')^2 = (1) - 2m\,(2) + m^2\,(3).$$

The eikonal is thus a function of α, β, γ; and if u be the form required to produce perfect definition for the conjugate planes m, it may be shown that u is a function of γ only.

§ 4. *The Aberration Function.*—Let $(u - \Phi)$ be the reduced eikonal when the base points are conjugate, and let u be the eikonal which would give perfect definition over the conjugate planes passing through the base points. Then $(-\Phi)$ involves the presence of aberrations and, therefore, Φ is known as the 'Aberration Function'; the negative sign is attached to Φ because $(-\Phi)$ corresponds to a function $(+\Psi)$ associated with the Characteristic Function, and, moreover, it ensures that, for standard cases, the coefficients in Φ shall be positive. If (Y, Z) (Y′, Z′) be the points in which a ray cuts the conjugate planes, then

$$\mathrm{Y}' - m\mathrm{Y} = -\left(\frac{\partial}{\partial \mathrm{M}'} + m\frac{\partial}{\partial \mathrm{M}}\right)u + \frac{\partial \Phi}{\partial \mathrm{M}'} + m\frac{\partial \Phi}{\partial \mathrm{M}},$$

i.e.,

$$\mathrm{Y}' - m\mathrm{Y} = \frac{\partial \Phi}{\partial \mathrm{M}'} + m\frac{\partial \Phi}{\partial \mathrm{M}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since u is a function of γ only; and similarly

$$\mathrm{Z}' - m\mathrm{Z} = \frac{\partial \Phi}{\partial \mathrm{N}'} + m\frac{\partial \Phi}{\partial \mathrm{N}}.$$

These formulæ give completely the aberration or wandering of the ray from the ideal focus. Φ is a function of α, β, γ, and therefore

$$\Phi = u_2 + u_3 + \ldots + u_n + \ldots$$

where u_n is a polynomial homogeneous and of degree n in α, β, γ; u_1 is omitted, since the aberrations are essentially of higher order than the first. The aberrations of the nth order will be said to be those given by the function u_n; it is convenient to assume that

$$u_2 = \tfrac{1}{8}\left[\sigma_1\alpha^2 - 4\sigma_2\alpha\beta + 2\sigma_3\alpha\gamma + 4\sigma_4\beta^2 - 4\sigma_5\beta\gamma + \sigma_6\gamma^2\right] . \quad . \quad . \quad . \quad (2)$$

and then $\sigma_1 \ldots \sigma_6$ are known as 'aberration coefficients.' The numerical factors which appear are chosen because of subsequent simplicity; they are suggested by the expansion of $[(1) - 2(2) + (3]^2$ which occurs in § 2.

§ 5. *The Addition of Aberrations.*—The reduced focal eikonal U for a single spherical surface has been given in § 2: let $(u - \Phi)$ be the reduced eikonal, the base points being conjugate for magnification m; then with the usual notation

$$u - \Phi = \mathrm{U} + \frac{1}{m}[1 - \mathrm{L}] + m[1 - \mathrm{L}'] . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The eikonal for a coaxial system is required; let this be $(u - \Phi)$ and the power of the system J : let suffix λ denote a similar function for the intermediate single surface λ. Then it may be shown that

$$\frac{u-\Phi}{J} = \sum_{\lambda=1}^{n} \left(\frac{u-\Phi}{J}\right)_\lambda, \quad \ldots \ldots \quad (2)$$

where the whole system is composed of surfaces 1 to λ inclusive. This is known as the 'equation for the addition of aberrations'; by means of it may be calculated the aberration coefficients for the whole system in terms of those for the intermediate single surfaces. In fact, it may be shown that

$$\left(\frac{\sigma_1}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_1}{J}\right)_\lambda m^4{}_{\lambda+1,n}, \quad \left(\frac{\sigma_2}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_2}{J}\right)_\lambda m^3{}_{\lambda+1,n}\, s_{\lambda+1,n}, \quad \left(\frac{\sigma_3}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_3}{J}\right)_\lambda m^2{}_{\lambda+1,n}\, s^2{}_{\lambda+1,n},$$

$$\left(\frac{\sigma_4}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_4}{J}\right)_\lambda m^2{}_{\lambda+1,n}\, s^2{}_{\lambda+1,n}, \quad \left(\frac{\sigma_5}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_5}{J}\right)_\lambda m^3{}_{\lambda+1,n}\, s_{\lambda+1,n}, \quad \left(\frac{\sigma_6}{J}\right)_{1,n} = \sum_{\lambda=1}^{n} \left(\frac{\sigma_6}{J}\right)_\lambda s^4{}_{\lambda+1,n}.$$

§ 6. *The First Order Aberrations.*—It may be shown that, if aberration be absent for the centre of the stop,

$$* \quad u = \frac{(s-m)^2}{2m}\gamma + \tfrac{1}{8}\frac{(s-m)(s^3-m)}{m}\gamma^2 + \ldots, \quad \ldots \ldots \quad (1)$$

so that with the usual notation, and neglecting aberrations,

$$Y = \frac{M - mM'}{m}, \qquad Z = \frac{N - mN'}{m}.$$

Similarly, if (ρ, ϕ) be the point in which the ray cuts the exit pupil—the initial line in these co-ordinates being parallel to the axes of Y, Y_1'—

$$\rho \cos\phi = M - sM', \qquad \rho \sin\phi = N - sN',$$

so that

$$\alpha\,(s-m)^2 = \rho^2, \qquad \beta\,(s-m)^2 = \rho Y' \cos\phi, \qquad \gamma\,(s-m)^2 = Y'^2,$$

where it is assumed that the Y-axis passes through the object point. There is no loss of generality in this.

From § 4 (1)

$$Y' - mY = \left(\frac{\partial}{\partial M'} + m\frac{\partial}{\partial M}\right)\Phi = \frac{1}{2(s-m)^3}\left[\sigma_1 S + m\sigma_2 C + m^2(\sigma_3 + 2\sigma_4) A + m^3\sigma_5 D\right],$$

$$Z' - mZ = \left(\frac{\partial}{\partial N'} + m\frac{\partial}{\partial N}\right)\Phi = \frac{1}{2(s-m)^3}\left[\sigma_1 S' + m\sigma_2 C' + m^2\sigma_4 A'\right], \quad \ldots \ldots \quad (2)$$

where

$$\begin{matrix} S \\ S' \end{matrix} \equiv -\rho^3 \begin{matrix} \cos\phi \\ \sin\phi \end{matrix}; \quad \begin{matrix} C \\ C' \end{matrix} \equiv \rho^2 Y \begin{matrix} (2+\cos 2\phi) \\ \sin 2\phi \end{matrix}; \quad \begin{matrix} A \\ A' \end{matrix} \equiv -\rho Y^2 \begin{matrix} \cos\phi \\ \sin\phi \end{matrix}; \quad D = Y^3.$$

§ 7. *Spherical Aberration.*—The first term in § 6 (2) shows that the ray cuts the object plane on the circumference of a circle of radius $\frac{\sigma_1}{2}\frac{\rho^3}{(s-m)^2}$ and centre at the Gaussian

* T. SMITH, 'Trans. Opt. Soc.,' vol. 23 (1921–22), No. 5.

image point : this is clearly 'spherical aberration' and the longitudinal aberration is $\frac{\sigma_1}{2} \frac{\rho^2}{(s-m)^2}$.

Coma.—The second term gives

$$Y' - mY = \frac{m\sigma_2}{2(s-m)^3} \rho^2 Y (2 + \cos 2\phi), \qquad Z' - mZ = \frac{m\sigma_2}{2(s-m)^3} \rho^2 Y \sin 2\phi.$$

This is the usual coma displacement ; the image ray cuts the object plane on the circumference of a circle of radius $\frac{m\sigma_2\rho^2 Y}{2(s-m)^3}$, whose centre is the point $\left(\frac{m\sigma_2\rho^2 Y}{(s-m)^3}, 0\right)$ referred to parallel axes through the Gaussian image. The conventional coma figure is, therefore, given subtending an angle of 60° at the Gaussian image point. It will be noticed that, owing to the appearance of the terms sin 2 θ, cos 2 θ, the coma-circle is described twice for every single description of the exit pupil.

Astigmatism and Curvature of the Field.—The third term gives

$$Y' - mY = -\frac{m^2(\sigma_3 + 2\sigma_4)}{2(s-m)^3} \rho Y^2 \cos\phi, \qquad Z' - mZ = -\frac{m^2\sigma_3}{(s-m)^3} \rho Y^2 \sin\phi.$$

This indicates that the image point lies upon one or other of two surfaces of revolution according as 'primary' or 'secondary' rays are considered. The curvatures of these surfaces $\left(\frac{1}{P_1}, \frac{1}{P_2}\right)$ are given by

$$\frac{1}{P_1} = \frac{\sigma_3 + 2\sigma_4}{(s-m)^2}, \quad \frac{1}{P_2} = \frac{\sigma_3}{(s-m)^2},$$

whence

$$\Delta\left(\frac{1}{P_1}\right) = \frac{3\sigma_4}{(s-m)^2} + \varpi, \qquad \Delta\left(\frac{1}{P_2}\right) = \frac{\sigma_4}{(s-m)^2} + \varpi, \quad . \quad . \quad . \quad . \quad (1)$$

where $\sigma_3 - \sigma_4 = (s-m)^2 \varpi$ and Δ is the usual operator of difference. These formulæ cover the case in which the incident beam is astigmatic. Thus, assuming a non-astigmatic incident beam, $\sigma_4 = 0$ denotes the co-incidence of the surfaces, *i.e.*, the absence of astigmatism, and then $\varpi = 0$ denotes the absence of curvature of this common surface. It may be shown readily that, for an air-air system ϖ (the Petzval sum) is given by the relation

$$\kappa_{1,n}\varpi = \sum_{\lambda=1}^{n} \frac{\kappa_\lambda}{\mu_{\lambda-1}\mu_\lambda}.$$

The formulæ (1) are identical with the formulæ usually given

$$\frac{1}{\mu'\rho_1'} - \frac{1}{\mu\rho_1} = \frac{1}{r}\left(\frac{1}{\mu} - \frac{1}{\mu'}\right) + 3\left(\frac{Q_x}{Q_x - Q_\xi}\right)^2 \left(\frac{1}{\mu'\xi'} - \frac{1}{\mu\xi}\right),$$

$$\frac{1}{\mu'\rho_2'} - \frac{1}{\mu\rho_2} = \frac{1}{r}\left(\frac{1}{\mu} - \frac{1}{\mu'}\right) + \left(\frac{Q_x}{Q_\xi - Q_x}\right)^2 \left(\frac{1}{\mu'\xi'} - \frac{1}{\mu\xi}\right),$$

cf. WHITTAKER, "The Symmetrical Optical Instrument," 'Camb. Math. Tracts.'

It is evident from (1) above that the distances of the primary and secondary foci from the Gaussian image plane are respectively

$$\frac{3\sigma_4 Y^2}{2(s-m)^2}+\frac{\varpi}{2}Y^2,\quad \frac{\sigma_4 Y^2}{2(s-m)^2}+\frac{\varpi}{2}Y^2,$$

or

$$\frac{Y^2}{2(s-m)^2}(\sigma_3+2\sigma_4),\quad \frac{Y^2\sigma_3}{2(s-m)^2}.$$

Distortion.—The last term gives

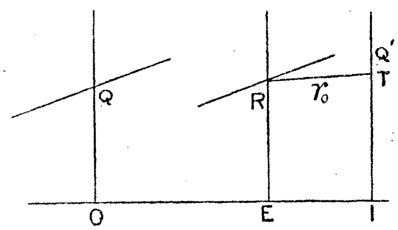
$$Y'-mY=\frac{\sigma_5}{2(s-m)^3}m^3Y^3,\qquad Z-mZ'=0.$$

This indicates a radial distortion; if $\sigma_5 > 0$ there is 'pin-cushion' distortion, while if $\sigma_5 < 0$ there is 'barrel' distortion.

From the above it has appeared that there are five aberrations of the first order—the five aberrations of VON SEIDEL—and that these aberrations are given completely by the five aberration coefficients σ_1, σ_2, σ_3, σ_4, σ_5.

PART II.

§ 1. With axes and notation as in Part I, let (L, M, N), (L′, M′, N′) respectively be the direction cosines of a ray before and after refraction through an optical system; let the ray meet the object plane O in Q and the exit pupil E in R, and let I and Q′ be the Gaussian conjugates to O and Q. Let U_1 be the modified optical distance from Q to R measured along the ray: the phase difference between the light vibrations at R and Q is $\frac{2\pi}{\lambda}U_1$, where λ is the wave-length of the light measured *in vacuo;* or, of course, the reduced length measured in any medium. It is to be noticed that λ is a modified length; it is multiplied by the modified power J of the optical system under consideration.



In the problem at present under discussion, a point source of light is considered; all the rays, therefore, will pass through Q and the actual phase at R may be taken to be $\frac{2\pi}{\lambda}U_1$. Let now T be a point near Q′ and in the plane I, and let RT $= r_0$ where, again, r_0 is modified and reduced; the exit pupil may be taken as a secondary surface of light disturbance and the phase contribution of the element of surface at R to the total disturbance at T is given by

$$\frac{2\pi}{\lambda}\left(U_1+r_0+\frac{\lambda}{4}\right).$$

If dS be an element of the exit pupil at R, the intensity of the disturbance at T is proportional to the squared modulus of the expression

$$\int e^{\frac{2\pi i}{\lambda}(U_1+r_0)}\,dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

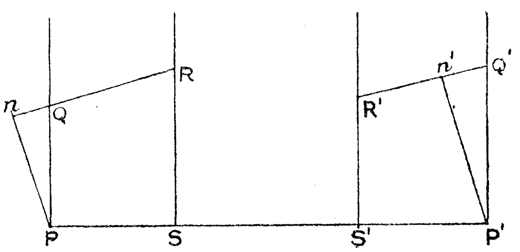
The amplitude of the light vibration has been omitted; this is justified by the considerations :—

(i) T is near Q′;

(ii) the wave surfaces are approximately concentric spheres with Q′ as centre. The amplitude will enter, therefore, as a factor of the form $\frac{A}{r_0}$, where A is a constant; and r_0 is, to the first order, equal to the distance between the planes E, I.

In conformity with the convention of Part I, U_1 may be considered to contain an additive constant which will ensure the vanishing of the function for an axial ray; and then $\frac{2\pi}{\lambda} U_1$ will be the difference in phase for the point R and the axial point of the exit pupil. Similarly, it is arranged that r_0 shall vanish when RT coincides with the axis of the optical system.

§2. Let P, P′ be a pair of conjugate points upon the axis of an optical system; and let S, S′ respectively be the axial points of the entrance and exit pupils. Let a ray whose direction cosines are (L, M, N), (L′, M′, N′) cut the conjugate planes through P, P′ in Q, Q′ respectively and the pupil planes in R, R′—the axes in the two spaces being chosen according to the usual conventions. Let Pn, P′n' be perpendicular to the ray. Let the co-ordinates of Q, Q′ be (Y, Z), (Y′, Z′), and let $(u - \Phi)$ be the reduced eikonal with base points P, P′; let U′ be the reduced modified distance from the object plane to the image plane measured along the ray.



Then nQ = MY + NZ, n'Q′ = M′Y′ + N′Z′, by projection of PQ upon nQ and of P′Q′ upon n'Q′; so that

$$U' = u - \Phi - (MY + NZ) + (M'Y' + N'Z')$$

$$= u - \Phi - \left(M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N} + M'\frac{\partial}{\partial M'} + N'\frac{\partial}{\partial N'}\right)(u - \Phi),$$

since

$$Y = \frac{\partial}{\partial M}(u - \Phi),\quad Y' = -\frac{\partial}{\partial M'}(u - \Phi),\ \&c., \&c.$$

If now $u = \sum_{n=1}^{\infty} v_{2n}$ and $\Phi = \sum_{n=2}^{\infty} \Phi_{2n}$ where v_{2n}, Φ_{2n} are each homogeneous and of degree $2n$ in M, N, M′ N′, then by substitution

$$U' = \sum_{n=1}^{\infty} v_{2n} - \sum_{n=2}^{\infty} \Phi_{2n} - \sum_{n=1}^{\infty} 2n v_{2n} + \sum_{n=2}^{\infty} 2n\Phi_{2n}$$

$$= -\sum_{n=1}^{\infty} (2n-1)\, v_{2n} + \sum_{n=2}^{\infty} (2n-1)\,\Phi_{2n}. \quad \ldots \ldots \quad (1)$$

Let $S'P' = d$; then the difference between $R'Q'$ and $S'P'$ is $\left(\frac{1}{L'} - 1\right) d$, and, therefore, if U_1 be the reduced modified distance from Q to R′

$$U_1 = U' + \left(1 - \frac{1}{L'}\right) d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From Part I

$$(3) = (M'^2 + N'^2)$$

so that

$$L' = [1 - (3)]^{\frac{1}{2}}$$

and

$$\frac{1}{L'} = [1 - (3)]^{-\frac{1}{2}} = 1 + \tfrac{1}{2}(3) + \tfrac{3}{8}(3)^2 + \ldots \quad \text{since} \quad |(3)| < 1.$$

Also

$(3) = \alpha - 2\beta + \gamma$, and, from the expansion for u (Part I, § 6),

$$u = \frac{(s-m)^2}{2m}\gamma + \frac{(s-m)(s^3-m)}{8m}\gamma^2 + \ldots,$$

so that

$$v_2 = \frac{(s-m)^2}{2m}\gamma, \quad v_4 = \frac{(s-m)(s^3-m)}{8m}\gamma^2.$$

Then from (1) and (2)

$$U_1 = -v_2 - 3v_4 + 3\Phi_4 - d\left\{\frac{\alpha - 2\beta + \gamma}{2} + \frac{3(\alpha - 2\beta + \gamma)^2}{8}\right\},$$

correct to the fourth order in the variables M, N, M′, N′; this is all that is necessary for a consideration of the first order aberrations. Substituting

$$U_1 = -\frac{(s-m)^2}{2m}\gamma - \frac{3(s-m)(s^3-m)}{8m}\gamma^2 + 3\Phi_4$$
$$- d\left\{\frac{\alpha - 2\beta + \gamma}{2} + \tfrac{3}{8}(\alpha - 2\beta + \gamma)^2\right\} \quad . \quad . \quad (3)$$

§ 3. Expression (3) gives U_1 in terms of the variables α, β, γ; in order to perform the integration of § 1 (1) it is necessary to obtain U_1 in terms of co-ordinates upon the exit pupil, and to this end α, β, γ must be expressed in such co-ordinates. In the customary notation

$$\alpha d^2 = (M - sM')^2 + (N - sN')^2$$
$$\beta d^2 = (M - sM')(M - mM') + (N - sN')(N - mN'),$$
$$\gamma d^2 = (M - mM')^2 + (N - mN')^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $Y_1 = mY = m\frac{\partial u}{\partial M} - m\frac{\partial \Phi}{\partial M}$ where (Y_1, Z_1) are the co-ordinates of the Gaussian image point. Since

$$u = \frac{(s-m)^2}{2m}\gamma + \frac{(s^3-m)(s-m)}{8m}\gamma^2 + \text{higher orders},$$

therefore

$$Y_1 = (M - mM')\left(1 + \tfrac{1}{2}\frac{s^3-m}{s-m}\gamma\right) - m\frac{\partial \Phi}{\partial M}, \quad \ldots \quad (2)$$

and, similarly,

$$\rho \cos\phi = (M - mM')\left(1 + \tfrac{1}{2}\frac{s^3-m}{s-m}\gamma\right) + \frac{\partial \Phi}{\partial M'} - M'd\left(1 + \frac{\alpha - 2\beta + \gamma}{2}\right).$$

Approximating now from (2)

$$M - mM' = Y_1; \; M - sM' = \rho\cos\phi, \text{ as a first approximation.}$$

Again, as a second approximation,

$$M - mM' = \left(Y_1 + m\frac{\partial \Phi_4}{\partial M}\right)\left(1 - \tfrac{1}{2}\frac{s^3-m}{s-m}\gamma\right)$$

$$= Y_1\left(1 - \tfrac{1}{2}\frac{s^3-m}{s-m}\gamma\right) + m\frac{\partial \Phi_4}{\partial M} + \text{higher orders},$$

and

$$M - sM' = \rho\cos\phi - \tfrac{1}{2}\frac{s^3-m}{s-m}\gamma\,(M - mM') - \frac{\partial \Phi_4}{\partial M'} + \frac{M'}{2}(\alpha - 2\beta + \gamma)\,d.$$

Applying now the first approximations to equations (1)

$$\alpha d^2 = \rho^2, \quad \beta d^2 = \rho\,(Y_1\cos\phi + Z_1\sin\phi), \quad \gamma d^2 = Y_1^2 + Z_1^2,$$

and using these results in the second approximation above

$$M - mM' = Y_1 - \tfrac{1}{2}\frac{s^3-m}{(s-m)^3}Y_1(Y_1^2 + Z_1^2) + m\frac{\partial \Phi_4}{\partial M} \quad \ldots \quad (3)$$

$$M - sM' = \rho\cos\phi - \tfrac{1}{2}\frac{s^3-m}{(s-m)^3}Y_1(Y_1^2 + Z_1^2) - \frac{\partial \Phi_4}{\partial M'}$$

$$+ \frac{M'}{2d}(\rho^2 - 2\rho Y_1\cos\phi - 2\rho Z_1\sin\phi + Y_1^2 + Z_1^2),$$

together with similar expressions for N, N′.

Second approximations can now be obtained for the variables α, β, γ; thus from (1) by means of (3) and assuming as usual that $Z_1 = 0$

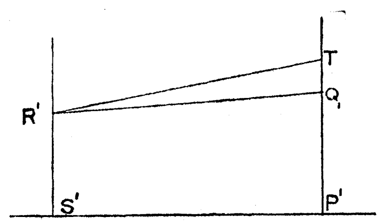
$$\alpha d^2 = \rho^2 - \frac{s^3 - m}{(s-m)^3}\rho Y_1^3 \cos\phi - 2\rho\left(\cos\phi \frac{\partial\Phi_4}{\partial M'} + \sin\phi \frac{\partial\Phi_4}{\partial N'}\right)$$
$$+ \frac{\rho}{d}(M'\cos\phi + N'\sin\phi)(\rho^2 - 2\rho Y_1 \cos\phi + Y_1^2)$$

$$\beta d^2 = \rho Y_1 \cos\phi\left(1 - \tfrac{1}{2}\frac{s^3 - m}{(s-m)^3} Y_1^2\right) - \tfrac{1}{2}\frac{s^3 - m}{(s-m)^3}Y_1^4 + m\rho\left(\cos\phi\frac{\partial\Phi_4}{\partial M} + \sin\phi\frac{\partial\Phi_4}{\partial N}\right)$$
$$- Y_1\frac{\partial\Phi_4}{\partial M'} + \frac{M'Y_1}{2d}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2)$$

$$\gamma d^2 = Y_1^2 - \frac{s^3 - m}{(s-m)^3}Y_1^4 + 2mY_1\frac{\partial\Phi_4}{\partial M}. \qquad (4)$$

These values may be substituted in § 3 (2) and will give U_1 in terms of the variables (ρ, ϕ), which are co-ordinates upon the exit pupil.

§ 4. Let R′ be a point (ρ, ϕ) upon the exit pupil; let Q_1 (Y_1, Z_1) be the Gaussian image, and let T be the point in the image plane at which the light intensity is required. Let the polar co-ordinates of T be (ρ', ϕ') referred to parallel axes through Q_1; so that the co-ordinates of T referred to the original axes of the image space are



$$Y_1 + \rho'\cos\phi', \quad Z_1 + \rho'\sin\phi'.$$

Let $RQ = r_0$ and $S'P' = d$, where S′, P′ are the axial points of the exit pupil and object plane. Then

$$r_0^2 = d^2 + (Y_1 - \rho\cos\phi + \rho'\cos\phi')^2 + (Z_1 - \rho\sin\phi + \rho'\sin\phi')^2$$

so that

$$r_0 = d\left[1 + \frac{(Y_1 - \rho\cos\phi)^2 + (Z_1 - \rho\sin\phi)^2 + 2\rho'\cos\phi'(Y_1 - \rho\cos\phi) + 2\rho'\sin\phi'(Z_1 - \rho\sin\phi) + \rho'^2}{2d^2}\right.$$
$$- \frac{1}{8d^4}\{(Y_1 - \rho\cos\phi)^4 + (Z_1 - \rho\sin\phi)^4 - 2(Y_1 - \rho\cos\phi)^2(Z_1 - \rho\sin\phi)^2$$
$$\left. + 4\rho'(\overline{Y_1 - \rho\cos\phi}^2 + \overline{Z_1 - \rho\sin\phi}^2)(\cos\phi'\overline{Y_1 - \rho\cos\phi} + \sin\phi'\overline{Z_1 - \rho\sin\phi})\}\right]$$

together with terms of higher orders.

§ 5. From §§ 2, 4

$$U_1 + r_0 = -\frac{(s-m)^2}{2m}\gamma - \frac{3(s-m)(s^3-m)}{8m}\gamma^2 + 3\Phi_4 - d\left\{\frac{\alpha - 2\beta + \gamma}{2} + \tfrac{3}{8}(\alpha - 2\beta + \gamma)^2\right\}$$
$$+ \frac{Y_1^2 - 2\rho Y_1\cos\phi + \rho^2 + 2\rho'\cos\phi'(Y_1 - \rho\cos\phi) + 2\rho'\sin\phi'(-\rho\sin\phi) + \rho'^2}{2d}$$
$$- \frac{1}{8d^3}\{(Y_1 - \rho\cos\phi)^4 + \rho^4\sin^4\phi + 2\rho^2\sin^2\phi(Y_1 - \rho\cos\phi)^2$$
$$+ 4\rho'(\overline{Y_1 - \rho\cos\phi}^2 + \rho^2\sin^2\phi)(\cos\phi'\overline{Y_1 - \rho\cos\phi} - \rho\sin\phi\cos\phi)\},$$

where the first term in the expansion for r_0 in § 4 has been omitted since r_0 must vanish for the axial ray. Applying now the approximations of § 3,

$$U_1 + r_0 = -\frac{1}{2m}\left(Y_1^2 - \frac{s^3-m}{s-m}Y_1^4 + 2mY_1\frac{\partial\Phi_4}{\partial M}\right) - \frac{3(s-m)(s^3-m)}{8md^4}Y_1^4$$
$$+ 3\Phi_4 - \frac{3}{8d^3}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2)^2$$
$$-\frac{1}{2d}\left\{\rho^2 - \frac{s^3-m}{(s-m)^3}\rho Y_1^3\cos\phi - 2\rho\left(\cos\phi\frac{\partial\Phi_4}{\partial M'} + \sin\phi\frac{\partial\Phi_4}{\partial N'}\right)\right.$$
$$+\frac{\rho}{d}(M'\cos\phi + N'\sin\phi)(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2) - 2\rho Y_1\cos\phi$$
$$+\frac{s^3-m}{(s-m)^3}\rho Y_1^3\cos\phi + \frac{s^3-m}{(s-m)^3}Y_1^4 - 2m\rho\left(\cos\phi\frac{\partial\Phi_4}{\partial M} + \sin\phi\frac{\partial\Phi_4}{\partial N}\right)$$
$$\left.+2Y_1\frac{\partial\Phi_4}{\partial M'} - \frac{M'Y_1}{d}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2) + Y_1^2 - \frac{s^3-m}{(s-m)^3}Y_1^4 + 2mY_1\frac{\partial\Phi_4}{\partial M}\right\}$$
$$+\frac{Y_1^2 - 2\rho Y_1\cos\phi + \rho^2 + 2\rho'\cos\phi'(Y_1 - \rho\cos\phi) + 2\rho'\sin\phi'(-\rho\sin\phi) + \rho'^2}{2d}$$
$$-\frac{1}{8d^3}\{(Y_1 - \rho\cos\phi)^4 + \rho^4\sin^4\phi + 2\rho^2\sin^2\phi\,(Y_1 - \rho\cos\phi)^2\},$$

where, in the last line, the terms in ρ' are omitted as being of higher order.

Now the coefficient of $\frac{\partial\Phi_4}{\partial M'}$, is $\left(\frac{\rho\cos\phi}{d} - \frac{Y_1}{d}\right)$, *i.e.* $(-M')$ approximately; and similarly the coefficient of $\left(-\frac{\partial\Phi_4}{\partial M}\right)$ is M, and, therefore, the terms in Φ_4 reduce to

$$-\left(M'\frac{\partial\Phi_4}{\partial M'} + N'\frac{\partial\Phi_4}{\partial N'} + M\frac{\partial\Phi_4}{\partial M} + N\frac{\partial\Phi_4}{\partial N}\right) = -4\Phi_4. \quad \quad (1)$$

Again, the second order terms are

$$-\frac{1}{2m}Y_1^2 - \frac{\rho^2}{2d} + \frac{\rho Y_1\cos\phi}{d} - \frac{Y_1^2}{2d} + \frac{Y_1^2}{2d} - \frac{\rho Y_1\cos\phi}{d} + \frac{\rho^2}{2d} = -\frac{1}{2m}Y_1^2. \quad \quad (2)$$

And

$$-\frac{\rho}{2d^2}(M'\cos\phi + N'\sin\phi)(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2) + \frac{M'Y}{2d^2}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2)$$
$$= -\frac{1}{2d^3}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2)(\overline{Y_1 - \rho\cos\phi}^2 + \rho^2\sin^2\phi)$$
$$= \frac{1}{2d^3}(\rho^2 - 2\rho Y_1\cos\phi + Y_1^2)^2, \quad \quad (3)$$

since $\rho\cos\phi = M - sM'$, $\rho\sin\phi = N - sN$ $Y_1 = M - mM'$, $O = N - mN'$.

This term (3) together with the term $-\frac{3}{8d^3}(\rho^2 - 2\rho Y_1 \cos\phi + Y_1^2)^2$, which appears in the expression for $U_1 + r_0$ given above, will cancel the last term, which is

$$-\frac{1}{8d^3}(Y_1^4 - 4Y_1^3\rho\cos\phi + 6Y_1^2\rho^2\cos^2\phi - 4Y_1\rho^3\cos^3\phi + \rho^4\cos^4\phi + \rho^4\sin^4\phi$$
$$+ 2\rho^2\sin^2\phi\,\overline{Y_1^2 - 2\rho Y_1\cos\phi + \rho^2\cos^2\phi})$$
$$= -\frac{1}{8d^3}(Y_1^4 - 4Y_1^3\rho\cos\phi + \overline{4\cos^2\phi + 2}\,Y_1^2\rho^2 - 4Y_1\rho^3\cos\phi + \rho^4)$$
$$= -\frac{1}{8d^3}(Y_1^2 - 2\rho Y_1\cos\phi + \rho^2)^2.$$

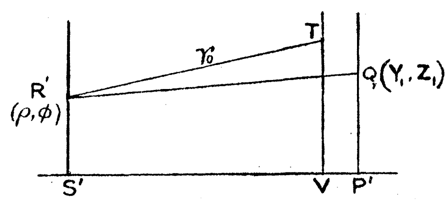
Neglecting, therefore, terms in ρ' the only remaining terms of the fourth order are

$$\frac{Y_1^4}{d^3}\left(\frac{s^3-m}{2m} - \frac{3}{8}\frac{s^3-m}{m}\right) \quad i.e., \quad \frac{Y_1^4(s^3-m)}{8md^3}, \qquad \text{. . . . . . (4)}$$

and, finally, collecting these results

$$U_1 + r_0 = -\frac{1}{2m}Y_1^2 + \frac{1}{8m}Y_1^4\frac{s^3-m}{d^3} - \Phi_4 + \frac{Y_1\rho'\cos\phi'}{d} - \frac{\rho\rho'}{d}\cos(\phi - \phi'). \; . \quad (5)$$

§ 6. An expression has now been obtained for $U_1 + r_0$ in terms of (ρ, ϕ), the variables of integration, but it has been assumed that the pole T (ρ', ϕ') is in the Gaussian image plane. The light distribution, however, is required not only in the Gaussian plane but in neighbouring planes also, and, therefore, to complete the investigation it is necessary to find the modification in the form of $U_1 + r_0$, so that these cases may be included. Let the notation be that of § 4 and let T be in a plane cutting the axis of the system at right angles in V; let $VP' = X$, so that X is considered positive when T departs from the Gaussian plane towards the exit pupil of the system. Let the co-ordinate of T be (ρ', ϕ') relative to the orthogonal projection of Q_1 upon the plane through V. Then



$$r_0^2 = (d - X)^2 + (Y_1 - \rho\cos\phi + \rho'\cos\phi')^2 + (Z_1 - \rho\sin\phi + \rho'\sin\phi')^2,$$

and comparing this with the expression given in § 4 it is seen that d has been replaced by $d - X$. Neglect terms in that paragraph which contain either ρ' or fourth-order terms, since in such the substitution of $(d - X)$ for d will make a change of higher order: the terms remaining are

$$\frac{Y_1^2 - 2\rho Y_1\cos\phi + \rho^2}{2d};$$

and the change occasioned in these terms is, to the degree of approximation contemplated,

$$\frac{X}{2d^2}(Y_1^2 - 2\rho Y_1 \cos\phi + \rho^2).$$

Adding this to § 5 (5) the general value of $U_1 + r_0$ is obtained, correct to the fourth order; it is

$$U_1 + r_0 = -\frac{1}{2m}Y_1^2 + \frac{1}{8m}Y_1^4\frac{s^3-m}{d^3} - \Phi_4 + \frac{Y_1\rho'}{d}\cos\phi' - \frac{\rho\rho'}{d}\cos(\phi-\phi')$$
$$+ \frac{X}{2d^2}(Y_1^2 + \rho^2 - 2Y_1\rho\cos\phi). \quad . \quad . \quad . \quad . \quad (1)$$

§ 7. *Spherical Aberration.*—If spherical aberration only be present then all the aberration coefficients will vanish, except σ_1; making these substitutions in the general integral, *viz.*, $\int e^{i\kappa(U_1+r_0)}d\text{S}$, we have the light intensity I at any point given by the formula

$$\sqrt{\text{I}} = \left|\int_0^1 e^{i\mu t + i\nu t^2} J_0(z\sqrt{t})\,dt\right|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the intensity at the Gaussian point is taken to be unity. Clearly the correct generalisation for spherical aberration of higher orders is

$$\sqrt{\text{I}} = \left|\int_0^1 e^{i\mu t + i\nu t^2 + i\xi t^3 + \dots} J_0(z\sqrt{t})\,dt\right|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

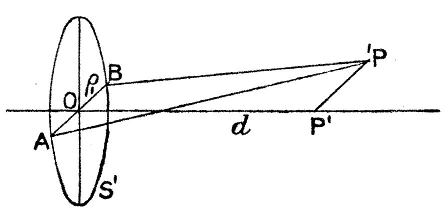
If aberrations be absent altogether, then, for the Gaussian plane, given by $\mu = 0$,

$$\sqrt{\text{I}} = \left|\int_0^1 J_0(z\sqrt{t})\,dt\right| = \left|\frac{2J_1(z)}{z}\right|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and this is the ordinary AIRY disc. For neighbouring planes

$$\sqrt{\text{I}} = \left|\int_0^1 e^{i\mu t} J_0(z\sqrt{t})\,dt\right|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

leading to ordinary Lommel functions [*cf.* Appendix § 4].



§ 8. Let S′ be the exit pupil of radius ρ_1 and let P be the pole at which the light intensity is required; let P be distant ρ' from the axis of the instrument, to which PP′ is perpendicular. Let AB be the diameter of S′ parallel to P′P.

Then $\text{AP} \sim \text{BP} = \frac{2\rho'\rho_1}{d}$, approximately, if ρ_1, ρ' be small compared with d; so that the difference in phase at P between rays from A, B is $2\frac{2\pi}{\lambda}\frac{\rho_1\rho'}{d}$, *i.e.*, $\frac{2\kappa\rho_1\rho'}{d}$ or $2z$. Thus, z may be used as a co-ordinate of P upon the image plane.

Moreover,

$$\kappa(U_1 + r_0) = \mu z^2 + \nu z^4 + \xi z^6 + \ldots$$
$$= 2\pi n + 2\pi n_1 + 2\pi n_2 + \ldots,$$

where $n_1, n_2, n_3, \ldots$, are the numbers of wave-lengths excess of phase of the extreme ray over the mean ray, due respectively to plane shift, first-order aberration, second-order aberration and so on.

§ 9. It is shown in the Appendix that the consideration of the general integral § 7(1) leads to two expressions for the light intensity which are convenient according as z is large or small. Thus, putting $n = 0$, $n_1 = 0$ in § 3 Appendix,

$U = \frac{2J_1(z)}{z}$, $V = 0$ and the light intensity is given by

$$I = U^2 + V^2 = 4\left[\frac{J_1(z)}{z}\right]^2 \quad . \quad . \quad . \quad . \quad (1)$$

The light contours are therefore concentric circles with centre at the Gaussian image point and the positions of the maxima are given by the roots of the equation

$$J_2(z) = 0. \quad . \quad . \quad . \quad . \quad (2)$$

This is the 'AIRY' ring system.

If $n_1 = 0$, *i.e.*, we consider only out-of-focus effects then substituting in § 3 Appendix, $u_2 = 2\pi n$, $\beta_2 = 0$; so that

$$c_{2r} = (-1)^r(2\pi n)^{2r}, \quad c_{2r+1} = 0, \quad d_{2r} = 0, \quad d_{2r+1} = (2\pi n)^{2r+1}(-1)^r,$$

and therefore

$$I = U^2 + V^2$$

where

$$2\pi n U = \frac{4\pi n}{z}J_1(z) - \left(\frac{4\pi n}{z}\right)^3 J_3(z) + \ldots$$

$$2\pi n V = -\left(\frac{4\pi n}{z}\right)^2 J_2(z) + \left(\frac{4\pi n}{z}\right)^4 J_4(z) - \ldots$$

and these are Lommel functions. Similarly, an expression may be obtained for the intensity in terms of Lommel functions of the second type and suitable for small values of z (Appendix, § 5).

It is especially useful to consider the intensity at points upon the axis of the optical system, and to this end the expression (2) in § 4 (Appendix) lends itself; for if $z = 0$

$$\sqrt{I} = |u_1|. \quad . \quad . \quad . \quad . \quad (3)$$

Now, in general,*

$$f = e^{i\mu x + i\nu x^2 + i\xi x^3 + \dots}$$

and

$$u_1 x = \int_0^x e^{i\mu t + i\nu t^2 + \dots}\, dt :$$

writing now $\mu t = v$, and remembering that $\mu x = 2\pi n$, $\nu x^2 = 2\pi n_1$, ...

$$u_1 = \frac{1}{2\pi n} \int_0^{2\pi n} e^{i\left(v + \frac{n_1}{2\pi n^2} v^2 + \frac{n_2}{4\pi^2 n^3} v^3 + \dots + \frac{n_s}{2^s \pi^s n^{s+1}} v^{s+1} + \dots\right)} dv, \quad . \quad . \quad . \quad . \quad (4)$$

where n_s refers to the aberration of order s. This integral, therefore, gives the light intensity at any point of the axis in the neighbourhood of the paraxial focus ; it assumes that n does not vanish. If, however, the lowest order aberration which appears be that of order s, then

$$u_1 = \left(\frac{1}{2\pi n_s}\right)^{\frac{1}{s+1}} \int_0^{2\pi n_s} e^{i\left(v^{s+1} + \frac{2\pi n_{s+1}}{(2\pi n_s)^{(s+2)/(s+1)}} v^{s+2} + \dots\right)} dv. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and this reduces to (4) when $s = 0$.

§ 10. (1) Let $n_s = 0$ where $s > 0$, *i.e.*, consider only out-of-focus effects in the absence of aberration. From § 9 (3)

$$u_1 = \frac{1}{2\pi n} \int_0^{2\pi n} e^{iv}\, dv$$

so that

$$\mathrm{I} = |u_1|^2 = \left(\frac{\sin \pi n}{\pi n}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

thus giving the known formula for the variation of intensity along the axis of the system ; it is seen that there are dark points given by

$$n = \text{an integer}.$$

(2) Let $n_s = 0$, $s > 1$, *i.e.*, consider the effect of first order aberration together with out-of-focus effects. Then

$$u_1 = \frac{1}{2\pi n} \int_0^{2\pi n} e^{i\left(v + \frac{n_1}{2\pi n^2} v^2\right)}\, dv ;$$

by a change of variable $\frac{\pi n^2}{2n_1} + v + \frac{n_1}{2\pi n^2} v^2 = \frac{\pi}{2} t^2$ this reduces to

$$\sqrt{\mathrm{I}} = |u_1| = \left| \frac{1}{2\sqrt{n_1}} \int_{\frac{n}{\sqrt{n_1}} + 2\sqrt{n_1}}^{\frac{n}{\sqrt{n_1}}} e^{\frac{i\pi t^2}{2}}\, dt \right| , \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the value of this may be found at once from tables of FRESNEL's integrals.

* So including spherical aberrations of all orders.

Actually, it is shown that

$$u_1 = \frac{e^{-i\frac{\pi}{2n_1}(n+2n_1)^2}}{2\sqrt{n_1}} \int_{\frac{n}{\sqrt{n_1}}+2\sqrt{n_1}}^{\frac{n}{\sqrt{n_1}}} e^{\frac{i\pi t^2}{2}}\, dt,$$

and it may be verified readily that this tends to equation (1) as n_1 tends to zero.

Let now $n_s = 0$ where $s > 2$, *i.e.*, introduce now second order aberration. Then

$$u_1 = \frac{1}{2\pi n}\int_0^{2\pi n} e^{i\left(v+\frac{n_1}{2\pi n^2}v^2+\frac{n_2}{2\pi n^3}v^3\right)}\, dv,$$

and this can be calculated by quadratures. To simplify the numerical work the term in v^2 may be eliminated by means of the substitution

$$\frac{n_1}{2\pi n^2}\left(v + \frac{2\pi}{3}\frac{nn_1}{3}\right)^3 = t^3,$$

and then

$$u_1 = \left(\frac{1}{4n_2}\right)^{1/3} \int_{\frac{n_1}{3n_2}(4n_2)^{1/3}}^{\left(1+\frac{n_1}{3n_2}\right)4n_2^{1/3}} e^{\frac{i\pi t^3}{2}+2i\pi t\left\{\frac{3nn_2-n_1^2}{3n_2(4n_2)^{1/3}}\right\}}\, dt. \quad . \quad . \qquad (3)$$

It is interesting to notice that if aberration of order s only be present, then at the Gaussian image point,

$$u_1 = \left(\frac{1}{4n_s}\right)^{\frac{1}{s+1}} \int_0^{(4n_s)^{\frac{1}{s+1}}} e^{\frac{i\pi}{2}u^{s+1}}\, du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

while for a neighbouring point upon the axis,

$$u_1 = \left(\frac{1}{4n_s}\right)^{\frac{1}{s+1}} \int_0^{(4n_s)^{\frac{1}{s+1}}} e^{\frac{i\pi}{2}\left(u^{s+1}+\frac{4n}{(4n_s)^{1/(s+1)}}u\right)}\, du. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

§ 11. *Some Numerical Results.*—(1) By substitution in § 10 (1), the following table is obtained :—

$$n = 0 : \quad \tfrac{1}{4} \quad : \quad \tfrac{1}{2} \quad : \quad \tfrac{3}{4} \quad : 1.$$
$$\mathrm{I} = 1 : 0\cdot8106 : 0\cdot4052 : 0\cdot091 : 0.$$

Thus if we adopt the Rayleigh limit, *i.e.*, assume that a 20 per cent. deterioration in the central intensity is the utmost which can be allowed, it is evident that $n = \frac{1}{4}$ brings us to this limit. The axial intensity is, of course, symmetrical about the point $n = 0$.

(2) Assume now the presence of first order aberration (n_1) only. By writing $n = -n_1 \pm \delta$, where δ is any quantity, in the integral § 10 (2), it is evident that the axial intensity is symmetrical about the point given by $n + n_1 = 0$, and this is the mid-point of the longitudinal aberration; for the marginal focus is given by $n + 2n_1 = 0$.*

* *Cf.* below § 12.

Let $n_1 = 1$; then from § 10 (2) the following table is obtained:

$$n = \quad 0 \quad : \quad -\tfrac{1}{2} \quad : \quad -1 \quad : \quad -\tfrac{3}{4} \quad : \quad -2 \quad : \quad -3.$$
$$I = 0 \cdot 0891 : 0 \cdot 3650 : 0 \cdot 8003 : 0 \cdot 3650 : 0 \cdot 0891 : 0 \cdot 0084.$$

The intensity at the paraxial focus falls, therefore, to about $0 \cdot 09$ when one wave-length of first order aberration is admitted; but by changing the focus to $n = -1$ this intensity rises to $0 \cdot 80$ and thus reaches the Rayleigh limit. This illustrates the great advantage effected by a slight change in focus. Outside the range included between the paraxial and marginal foci the intensity falls rapidly, but it is worthy of notice that there are now no dark points upon the axis.

(3) Let $n_1 = 0$, $n_s = 0$ where $s > 2$, *i.e.*, consider the effect of second order aberration in the absence of that of the first order. From § 10 (5)

$$u_1 = \frac{1}{(4n_2)^{1/3}} \int_0^{(4n_2)^{1/3}} e^{\frac{i\pi}{2}\left(u^3 + \frac{4n}{(4n_2)^{1/3}} u\right)} du;$$

the paraxial focus is given by $n = 0$ and the marginal focus by $n + 3n_2 = 0$,* and, assuming $n_2 = \frac{1}{2}$, the range of values of n given by $0 > n > -\frac{3}{2}$ has to be considered. By quadratures I find the following :—

$$2n = \quad 0 \quad : \quad 1 \quad : \quad 2.$$
$$I = 0 \cdot 4631 : 0 \cdot 8720 : 0 \cdot 3811.$$

Thus the maximum intensity occurs here at less than a third of the distance from the paraxial to the marginal focus, instead of mid-way as in the case of first-order aberrations; and it is evident that a little more only than half a wave-length of aberration can be admitted in order to satisfy the Rayleigh condition. This, however, is in the absence of first-order aberration; if the latter be present as much as four wave-lengths of second order aberration may be admitted. This is apparent from the following.

(4) The case of practical importance arises when aberrations of all orders are present; for it is usual to design a lens so that the paraxial and marginal foci coincide at full aperture. The marginal focus is defined by

$$\sum_{s=0}^{\infty} (s+1)\, n_s = 0,$$

so that

$$\sum_{s=1}^{\infty} (s+1)\, n_s = 0,$$

if the foci coincide at full aperture.

* *Cf.* below § 12.

The simplest case in which this can occur is that in which aberrations of the first two orders only are present; then $2n_1 + 3n_2 = 0$. Accordingly I have considered the two cases (1) $n_1 = -3$, $n_2 = 2$ and (2) $n_1 = -6$, $n_2 = 4$. From § 10 (5) the values of u_1 for these two cases are

$$(1) \quad u_1 = \int_0^1 \cos\frac{\pi}{2}\{u^3 + (2n-3)u\}\, du,$$

$$(2) \quad u_1 = \frac{1}{2^{\frac{1}{3}}}\int_0^1 \cos\frac{\pi}{2}\{u^3 + 4^{\frac{1}{3}}(n-3)u\}\, du.$$

Quadrature leads to the following table:

(1) $n =$ 0 : 1 : 5/4 : 3/2 : 2
$\mathrm{I} =$ 0·0776 : 0·8244 : 0·9378 : 0·7093 : 0·0912.

(2) $n =$ 2 : $2\frac{1}{4}$: $2\frac{1}{2}$: 3
$\mathrm{I} =$ 0·4674 : 0·7336 : 0·7778 : 0·2870.

Thus, in these circumstances, four wave-lengths of second-order aberration can be tolerated. It is interesting to investigate closing down the aperture in case (2): let the exit pupil be decreased to $\frac{1}{\sqrt{2}}$ of its full radius; this is equivalent to changing the values of n, n_1, n_2 in the following manner; n is divided by 2, n_1 by 4 and n_2 by 8. Thus

$$n_1 = -\tfrac{3}{2}, \quad n_2 = \tfrac{1}{2}, \quad \text{and} \quad u_1 = \frac{1}{2^{1/3}}\int_0^{2^{1/3}} e^{\frac{i\pi}{2}(u^3+4^{1/3}(2n-3)u)}\, du,$$

and quadrature gives

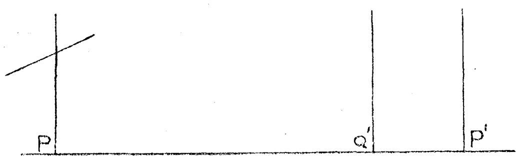
$n =$ 3 : $2\frac{1}{2}$: $2\frac{1}{4}$: 2
$\mathrm{I} =$ 0·4502 : 0·7788 : 0·8660 : 0·9098.

It must be noticed that 'n' in this table is equal to '$2n$' in the integral; and this is written so in order to compare the results with those given above. The actual intensities in the stopped down case must of course be divided by two in order to compare them with the full aperture case.

It is interesting to notice how the position of maximum intensity has moved: with full aperture it is given by $n = 2\frac{3}{8}$ approximately; with aperture stopped down it is given by $n < 2$.

Any other case of a combination of aberrations with the exit pupil stopped down can be treated similarly by means of § 10 (5).

§ 12. *Formulæ defining marginal focus.*—Let PP′ be axial conjugates for a symmetrical system and let Q′ be an axial point near to P′. Let $(u - \Phi)$ be the reduced eikonal for PP′; then if F be the reduced eikonal for the point P, Q′ we have



$$F = u - \Phi + A\alpha,$$

where A is a constant and α has its meaning as in Part I: the term $A\alpha$ is introduced because of the change of base point from P′ to Q′. Now if the marginal ray pass through Q′ we must have $\frac{\partial F}{\partial M'} = 0$,

$$i.e., \quad \frac{\partial}{\partial M'}(A\alpha - \Phi) = 0,$$

since, for rays starting from P

$$\frac{\partial u}{\partial M'} = m\frac{\partial u}{\partial M} = 0.$$

Thus if $\Phi = -\sum_{r=2}^{\infty} A_r\alpha^r$, we have,

$$A + 2A_2\alpha + 3A_3\alpha^2 + \ldots + rA_r\alpha^{r-1} + \ldots = 0,$$
$$i.e., \quad A\alpha + 2A_2\alpha^2 + \ldots + rA_r\alpha^r + \ldots = 0,$$

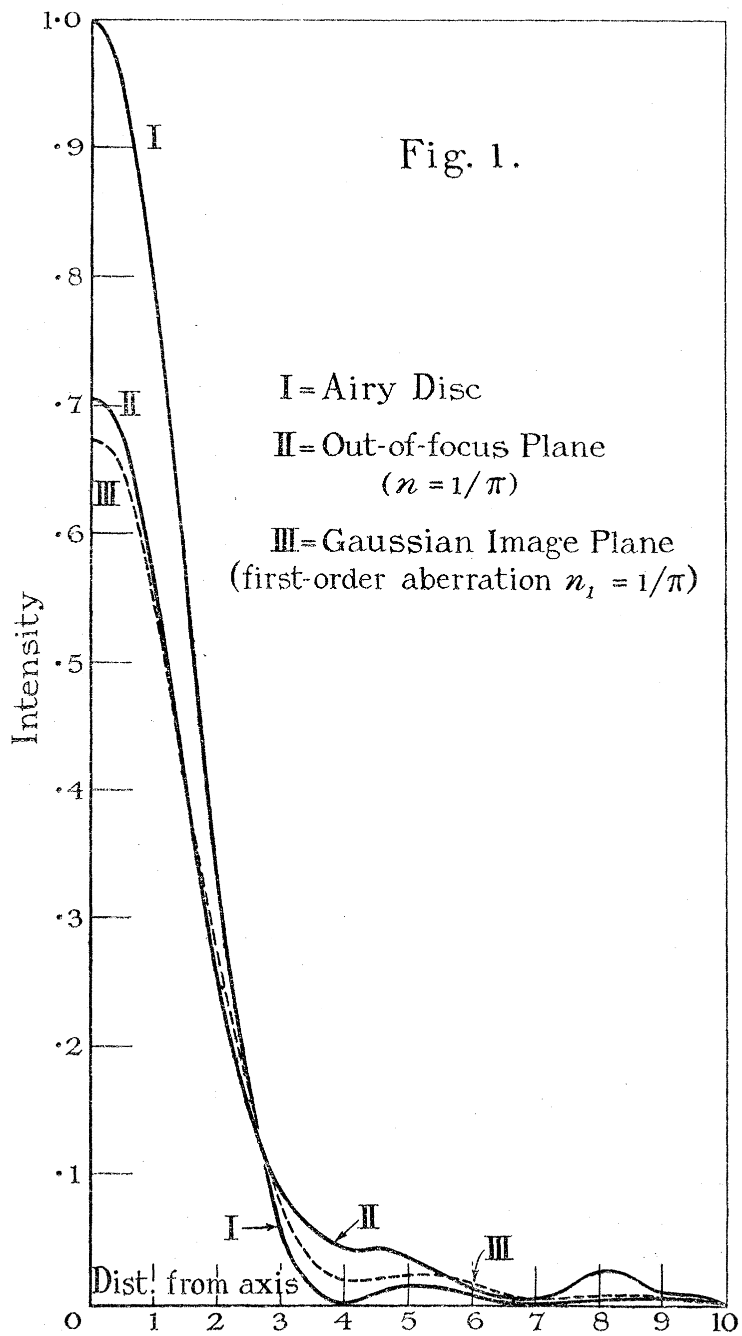
so that with the usual definitions of $n_1, n_2, \ldots n_r \ldots$

$$n + 2n_1 + 3n_2 + \ldots + (\overline{s+1})n_s + \ldots = 0,$$

giving the position (n) of the marginal focus in the presence of various orders of spherical aberration.

§ 13. A diagram is appended (fig. 1) showing the light intensity in three cases: (1) the Airy disc, *i.e.*, the diffraction pattern for a perfect instrument on the Gaussian image plane; (2) the pattern in out-of-focus condition simply, the change of focus being given by $n = 1/\pi$; and (3) the pattern upon the Gaussian image plane in the presence of spherical aberration of the first order of amount $n_1 = 1/\pi$. The particular values of n, n_1 are chosen to simplify the calculations and the formulæ used are those given in the text. The general effect is seen to be to scatter the light into the outer parts of the field; the light contours are, of course, in every case circles, and the diagram shows a half-section passing through the centre of the system of contours. The numerical results, for the intensity, are as follows:—

I	$z = 0$	1	2	3	4	5	6	7	8	9	10
Airy disc	1·000	0·775	0·333	0·051	0·001	0·017	0·008	0·000	0·003	0·003	0·000
$n = 1/\pi$	0·708	0·549	0·247	0·088	0·044	0·037	0·014	0·003	0·025	0·004	0·001
$n_1 = 1/\pi$	0·676	0·536	0·259	0·072	0·021	0·022	0·017	0·002	0·005	0·003	0·001



§ 14. *Coma.*—If coma be the only aberration present, then all the aberration coefficients vanish except σ_2 and the intensity is given by the squared modulus of the expression

$$\int_0^1 e^{i\mu v} J_0 \left(\sqrt{(\beta v - \varepsilon)^2 v - 2zv(\beta v - \varepsilon)\cos\phi' + z^2 v}\right) dv, \quad . \quad (1)\ [\text{App., § 6 (3)}]$$

where

$$zd = \kappa\rho_1\rho', \ 2\beta d^4 = \kappa\rho_1^3\sigma_2 Y_1, \ zd\sqrt{v} = \kappa\rho\rho', \ \varepsilon d^2 = \kappa\rho_1 XY_1, \ 2\mu d^2 = \kappa X\rho_1^2.$$

This is the general expression for the intensity in the presence of coma, whether in the paraxial image plane or not ; μ has the same meaning as before, and the geometrical significance of β is given below. If we consider only points upon the axis of the coma-figure, *i.e.*, put $\phi' = 0$, then (1) takes the form

$$\int_0^1 e^{i\mu v} J_0 \left(\beta v^{3/2} - \overline{z + \varepsilon} \, . \, v^{1/2}\right) dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

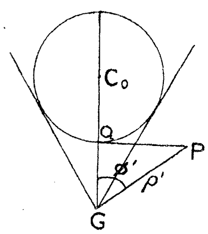
i.e.,

$$\int_0^1 e^{i\mu v} J_0 \left(\beta v^{3/2} - z' v^{1/2}\right) dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $z' = z + \varepsilon$ and therefore z' measures distance from the central line or the line joining the centre of the exit pupil to the Gaussian image point. It is evident now that a change in the sign of X, *i.e.*, of μ, leaves unaltered the modulus of (3), so that nothing is gained here by 'change of focus,' and the effects upon the Gaussian image plane may be considered to be the most important. We study, therefore, the case $X = 0$; and from (1) the intensity upon the Gaussian image plane is given by

$$\sqrt{I} = \int_0^1 J_0 \left(\sqrt{z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3}\right) dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Consider the geometrical coma figure. Let G be the Gaussian image, and the circle centre C_0 the coma-circle corresponding to an exit-pupil of radius ρ_1. Let P be the point (ρ', ϕ') and let GC_0 cut the circle in Q. Then from Part I



$$GQ = QC_0 = \frac{\sigma_2}{2} \frac{\rho_1{}^2 Y_1}{d^3},$$

all distances being modified and reduced. Thus

$$\beta = \frac{\kappa\rho_1}{d} \frac{\sigma_2}{2} \frac{\rho_1{}^2 Y_1}{d^3} = \frac{\kappa\rho_1}{d} GQ$$

will give the length GQ measured in the co-ordinates usual in this investigation; and

$$PQ^2 = z^2 - 2z\beta\cos\phi' + \beta^2,$$
$$z = GP.$$

Though not of much practical importance, it is interesting to observe that the series Appendix § 7 give at once two expressions for the intensity of illumination at the Gaussian image G in the presence of coma. The expressions are

$$\sqrt{I} = \sum_{n=0}^{\infty} \left(\frac{3\beta}{2}\right)^n \frac{J_n(\beta)}{1 \, . \, 4 \ldots 3n + 1},$$

$$\sqrt{I} = \sum_{n=1}^{\infty} \left(\frac{2}{\beta}\right)^n J_n(\beta) \, . \, \frac{1 \, . \, 2 \ldots 3n - 4}{3^n}.$$

§ 15. *The Light Contours.*—The image plane may be mapped out by drawing the curves of equal intensity; these will be the light contours. From § 14 (4) their equation, in polar co-ordinates, is

$$\int_0^1 J_0 \left(\sqrt{z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3}\right) dv = \sqrt{I} = \text{const.}, \quad . \quad . \quad . \quad . \quad (1)$$

or, in Cartesian co-ordinates,

$$\int_0^1 J_0\left(\sqrt{(x^2+y^2)\,v - 2y\beta v^2 + \beta^2 v^3}\right) = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These have been drawn for the particular case $\beta = 4$, and are given below.* If $\beta = 0$ then the contours are given by

$$\frac{J_0\left[(x^2+y^2)^{\frac{1}{2}}\right]}{(x^2+y^2)^{\frac{1}{2}}} = \text{const.},$$

and are therefore circles with common centre at the Gaussian image. If $\beta \neq 0$ then, in the outer parts of the field, the contours are given by the first term in the series of Appendix § 7, so that their equation is

$$\frac{2u_1 J_1(u_1)}{z^2 - 4\beta z \cos\phi' + 3\beta^2} = \text{const.},$$

and, if z be large compared with β, the contours are given approximately by neglecting β, *i.e*, by

$$\frac{2J_1(u_1)}{u_1} = \text{const.};$$

and, again, the AIRY disc is reproduced, but the common centre is now at the point $(\beta, 0)$. Between two consecutive dark 'circles' will be a band of illumination; the intensity will be symmetrical as usual about the line $\phi' = 0$, the axis of the coma figure, but from the formula

$$\sqrt{I} = \frac{2u_1 J_1(u_1)}{u_1{}^2 - 2\beta z \cos\phi' + 2\beta^2},$$

which is the first term of the series in Appendix § 7—an approximation for the outer parts of the field—it is evident that the intensity in the direction $\phi' = 0$ will be greater than the corresponding intensity in the direction $\phi' = \pi$. The light, therefore, is scattered in the direction away from the head of the 'coma figure' in contradiction to the result obtained from the geometrical theory.

§ 16. It is important to find the intensity upon the axis of the coma figure; put $\phi' = 0$ in § 14 (4). Then

$$\sqrt{I} = \int_0^1 J_0\left(\beta\sqrt{v^3} - z\sqrt{v}\right) dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and in this expression z may carry its own sign to cover the case of the line $\phi' = \pi$. Dark points occur upon the axis of the figure, and their positions are given by the roots of the equation in z

$$\int_0^1 J_0\left(\beta\sqrt{v^3} - z\sqrt{v}\right) dv = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The positions of the turning values of $\sqrt{I}$ are given, therefore, by the roots of the equation in z

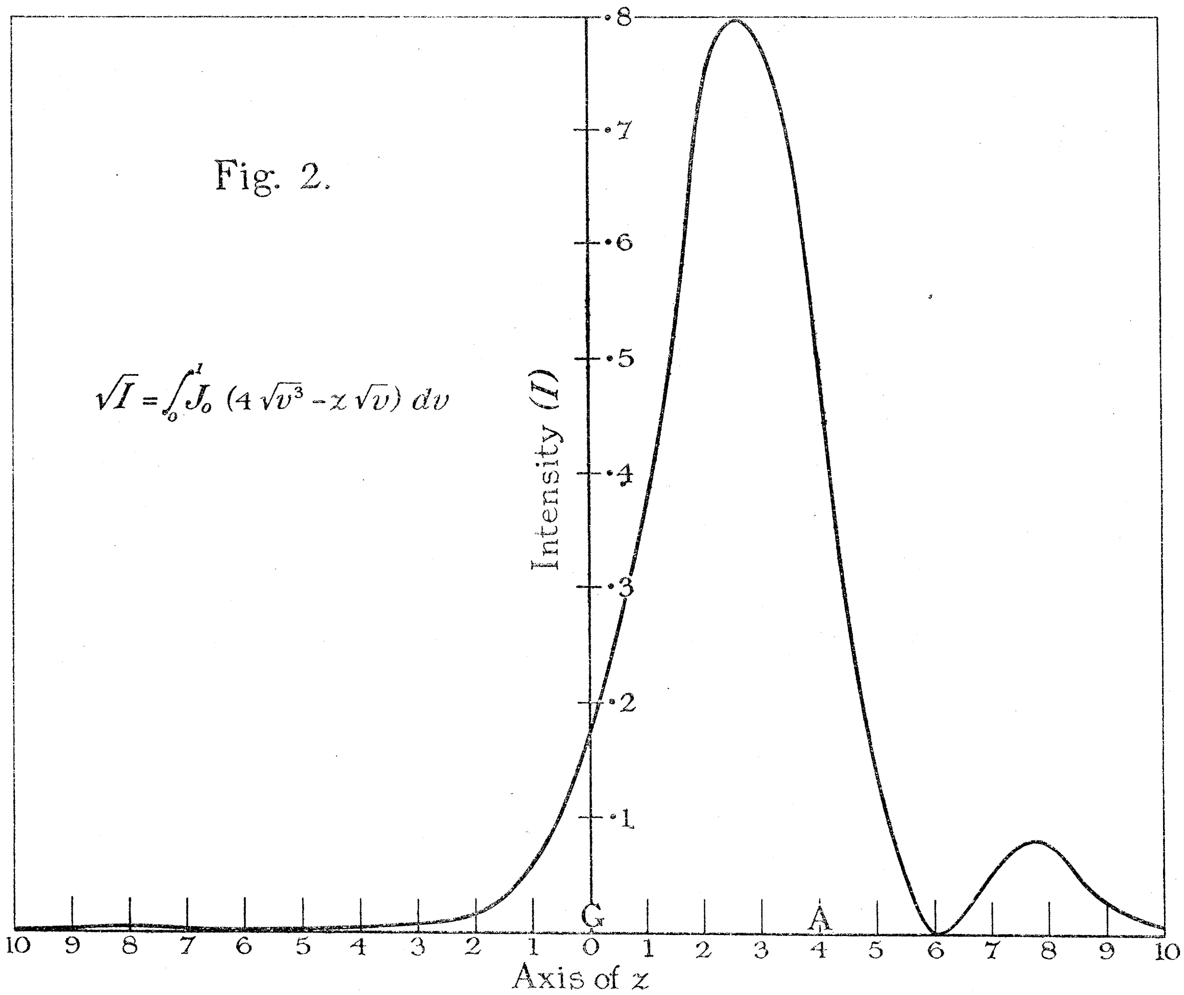
$$\int_0^1 J_0\left(\beta\sqrt{v^3} - z\sqrt{v}\right)\sqrt{v}\, dv = 0,$$

i.e.,

$$\int_0^1 J_1\left(\beta v - z\sqrt{v^3}\right) dv = 0.$$

* In § 18.

The roots of this equation will give, in general, the positions of the maxima of I, for $\sqrt{\mathrm{I}}$, as given by equation (1), has its maxima and minima upon opposite sides of the axis of z in general. A minimum value of I is indicated, however, if the numerically least negative root of (3) is less than the numerically least negative root of equation (1). This happens in the case $\beta = 3$.



Light Intensity along Axis of Coma Figure in presence of Coma of amount $\beta = 4$.

The numerical values of the intensities, which are shown graphically in Fig. 2, are as follows :—

z	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
Intensity	0·0001	0·0000	0·0002	0·0001	0·0004	0·0029	0·0054	0·0068	0·0150	0·0512

z	0	1	2	3	4	5	6	7	8	9	10.
Intensity	0·1749	0·3700	0·7286	0·7748	0·4931	0·1371	0·0003	0·0513	0·0763	0·0268	0·0056.

$z = 0$ is the point G the Gaussian Image Point.
$z = 4$ is the point Q of § 14.

The maximum value of the intensity is 0·80 (approximately) and occurs at the point $z = 2·6$.

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§ 17. The roots of (3), of the preceding paragraph, give the positions of the maxima and minima of the intensity upon the axis of the coma figure: when drawing the contours it is necessary to know whether these are positions of maxima and minima in two dimensions. To this end § 14 (4) lends itself.

Let

$$F(x, y) \equiv \sqrt{I} = \int_0^1 J_0 \sqrt{(x^2+y^2)\,v - 2y\beta v^2 + \beta^2 v^3}\, dv.$$

If (h, k) be a small variation in the image plane, then

$$F(x+h, y+k) = F(x, y) + \frac{1}{\underline{|1}}\left(h\frac{\partial F}{\partial x} + k\frac{\partial F}{\partial y}\right) + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 F}{\partial x^2} + 2hk\frac{\partial^2 F}{\partial x\,\partial y} + k^2\frac{\partial^2 F}{\partial y^2}\right) + \ldots;$$

and if two contours cross, as at a 'saddle-point,' their directions are given by

$$\frac{h}{k} = -\frac{\partial^2 F}{\partial x\,\partial y} \pm \sqrt{\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2}}.$$

Now upon the axis of y,

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = -\int_0^1 J_1(u)\sqrt{v}\, dv,$$

$$\frac{\partial^2 F}{\partial x^2} = -\int_0^1 \frac{J_1(u)}{u} v\, dv, \quad \frac{\partial^2 F}{\partial x\,\partial y} = 0, \quad \frac{\partial^2 F}{\partial y^2} = -\int_0^1 \frac{J_1(u)}{u}\, dv + \int_0^1 J_2(u)\, v\, dv,$$

where $u = y\sqrt{v} - \beta\sqrt{v^3}$.

Thus for true maxima and minima $\frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}$ must have the same sign at the points given by $\frac{\partial F}{\partial y} = 0$; *i.e.*,

$$\int_0^1 \{J_0(u) \pm J_2(u)\}\, v\, dv$$

must have the same sign if

$$\int_0^1 J_1(u)\sqrt{v} = 0;$$

otherwise the point will be a 'saddle-point.' The condition may be expressed as follows: maxima or minima do or do not occur according as

$$\left|\int_0^1 J_0(u)\, v\, dv\right| \gtrless \left|\int_0^1 J_2(u)\, v\, dv\right|,$$

where

$$\int_0^1 J_1(u)\sqrt{v}\, dv = 0 \qquad \text{and} \qquad u = y\sqrt{v} - \beta\sqrt{v^3}.$$

§ 18. Some numerical results are attached for the case in which $\beta = 4$. They are shown in the diagram, fig. 3. It will be seen that the central contours of the first 'dark ring' are very nearly circles, but that the outer contours are elongated in the negative direction of the axis of the coma figure; the central maximum is at a point M upon the axis of the figure (§ 14) such that GM = 2MQ very nearly, and the intensity there is 0·80 of the theoretical intensity at G in the absence of coma. This amount of coma, therefore, brings us to the RAYLEIGH limit. As the amount of coma is increased the central intensity point M moves relatively towards the head G of the coma-figure, until, when $\beta = 12$, M is very nearly midway between G and A, being slightly nearer G than A. The contours depart now very much more from the circular form.

Light Contours upon the Gaussian Image Plane in the presence of Coma of amount $\beta = 4$.

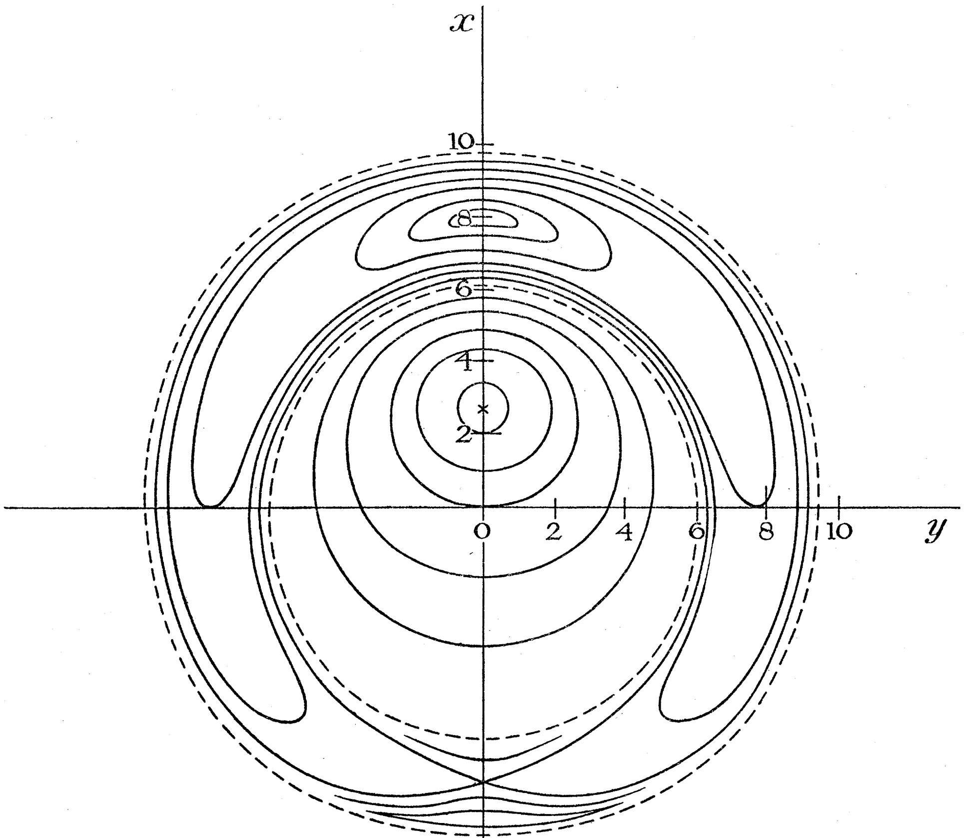


Fig. 3.

Dotted lines indicate zero contours.

Equation of Contours $\int_0^1 J_0 (\sqrt{x^2 + y^2 v - 8yv^2 + 16v^3})\, dv = \text{const.}$

It is evident from the diagrams that the light is scattered in a direction from the head of the coma figure; the intensity in this direction is considerably greater than that in the opposite direction, and, moreover, the intensity gradient is much greater too. This is the well-known ' comet ' appearance produced by coma. The general conclusion may be drawn that there is no similarity whatever between the actual (diffraction) coma figure and that indicated by geometrical theory; the latter shows a great intensity near the Gaussian image point within the angle marked, while diffraction theory leads to an entirely different conclusion. It has been pointed out already that the outer parts of the field approximate to the AIRY disc.

§ 19. Approximate roots of the equation

$$\int_0^1 J_1 (4v - zv^{1/3})\, dv = 0$$

are $z = 11 \cdot 4$, $7 \cdot 7$, $2 \cdot 7$, $-7 \cdot 7$. The 'radius' of the first light 'ring' is therefore $10 \cdot 4$ in the direction MG, and $5 \cdot 0$ in the opposite direction; while in the AIRY disc the radius in either direction is $5 \cdot 1$. The distortion effect of coma upon the first bright ring is thus very marked. The radii of the first dark 'ring' are respectively $9 \cdot 2$ and $3 \cdot 3$, compared with the AIRY disc $3 \cdot 3$; while those of the second 'ring' are 12 and 7, compared with 7 in the case of the AIRY disc. These again show central distortion which tends to disappear in the outer parts of the field.

It is to be noticed that in certain cases an incomplete 'ring' may appear; *e.g.*, in the case $\beta = 3$, a root of

$$\int_0^1 J_1 (3v - zv^{1/3})\, dv = 0$$

occurs at $z = -3 \cdot 5$ approximately; this, however, is a minimum value of the intensity other than zero, and as no such minimum value is given for positive values of z a partial 'ring' is formed—not completely surrounding the central maximum. In the case $\beta = 4$, this has disappeared and an almost stationary value of the intensity is obtained.

§ 20. It is useful to interpret these results in terms of the departure from the sine-condition. Let the paraxial magnification produced by an optical instrument be m; let the magnification calculated by the sine-condition applied to a marginal ray be $m (1 + \varepsilon)$. Then if Y_1 denote the distance of G (§ 14) from the axis of the system

$$\varepsilon Y_1 = GQ \quad \ldots \ldots \ldots \ldots \quad (1)$$

(see paragraph following for proof).

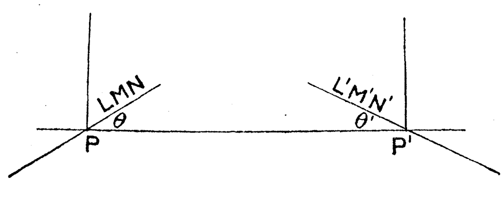
Now the Rayleigh limit is reached when β ($\equiv$ GQ) $= 4$, so that in this case

$$\varepsilon Y_1 = 4,$$

and Y_1 is measured in the z co-ordinates.

Thus in the case of a telescope objective it would appear that $\varepsilon = \frac{1}{300}$ would give an ample field in which the Rayleigh limit is not exceeded; this then gives the permissible departure from the sine-condition in this particular case.

§ 21. *The Sine-Condition.*—Let the diagram represent a symmetrical optical instrument with axis PP′ and let P, P′ be Gaussian conjugates. Let (L, M, N) be the direction cosines of a ray passing through P and inclined at an angle θ with PP′; if spherical aberration be absent this ray will pass through P′ after refraction: let (L′, M′, N′) be its direction cosines and θ′



the angle which it makes with the axis. Let m be the Gaussian magnification associated with the points PP′. Then in the customary notation

$\cos\theta = L$, $\cos\theta' = L'$; so that

$$\frac{\sin^2\theta'}{\sin^2\theta} = \frac{M'^2 + N'^2}{M^2 + N^2} = \frac{\alpha - 2\beta + \gamma}{m^2\alpha - 2sm\beta + s^2\gamma},$$

$$i.e., \quad \left(\frac{m\sin\theta'}{\sin\theta}\right)^2 = \frac{\alpha - 2\beta + \gamma}{\alpha - 2\frac{s}{m}\beta + \left(\frac{s}{m}\right)^2\gamma}.$$

Now since P, P′ are upon the axis, *i.e.*, $Y_1 = 0$, it follows that

$$\alpha d^2 = \rho^2 - 2\rho\left[\cos\phi\frac{\partial\Phi}{\partial M'} + \sin\phi\frac{\partial\Phi}{\partial N'}\right] + \rho^2\left[\frac{\rho\cos\phi}{d}M' + \frac{\rho\sin\phi}{d}N'\right],$$

$$\beta d^2 = m\rho\left[\cos\phi\frac{\partial\Phi}{\partial M} + \sin\phi\frac{\partial\Phi}{\partial N}\right],$$

$$\gamma d^2 = 0.$$

And

$$M - mM' = 0, \quad N - mN' = 0, \quad M - sM' = \rho\cos\phi, \quad N - sN' = \rho\sin\phi \quad . \quad . \quad . \quad (A)$$

approximately; so that

$$M'd + \rho\cos\phi = 0$$
$$N'd + \rho\sin\phi = 0 \quad \text{since} \quad d = s - m.$$

Then by substitution—

$$\alpha d^2 = \rho^2 + 2d\left[M'\frac{\partial\Phi}{\partial M'} + N'\frac{\partial\Phi}{\partial N'}\right] - \rho^2(M'^2 + N'^2) = \rho^2 L'^2 + 2d\left[M'\frac{\partial\Phi}{\partial M'} + N'\frac{\partial\Phi}{\partial N'}\right],$$

$$\beta d^2 = \quad -md\left[M'\frac{\partial\Phi}{\partial M} + N'\frac{\partial\Phi}{\partial N}\right] = -d\left[M\frac{\partial\Phi}{\partial M} + N\frac{\partial\Phi}{\partial N}\right]:$$

therefore

$$\left(m\frac{\sin\theta'}{\sin\theta}\right)^2 = \frac{\alpha - 2\beta}{\alpha - 2\frac{s}{m}\beta}$$

$$= \frac{\rho^2 L'^2 + 2d\left(M'\frac{\partial}{\partial M'} + N'\frac{\partial}{\partial N'} + M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N}\right)\Phi}{\rho^2 L'^2 + 2d\left(M'\frac{\partial}{\partial M'} + N'\frac{\partial}{\partial N'}\right)\Phi + 2\frac{s}{m}d\left(M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N}\right)\Phi}$$

$$= \frac{\rho^2 L'^2 + 2d\left(M'\frac{\partial}{\partial M'} + N'\frac{\partial}{\partial N'} + M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N}\right)\Phi}{\rho^2 L'^2 + 2d\left(M'\frac{\partial}{\partial M'} + N'\frac{\partial}{\partial N'} + M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N}\right)\Phi + 2\frac{d^2}{m}\left(M\frac{\partial}{\partial M} + N\frac{\partial}{\partial N}\right)}$$

$$= \frac{\rho^2 L'^2 + 8d\Phi}{\rho^2 L'^2 + 8d\Phi + \frac{2d^2}{m}\left(M\frac{\partial\Phi}{\partial M} + N\frac{\partial\Phi}{\partial N}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since Φ is homogeneous and of degree four in M, N, M′, N′, if we consider only the first order aberrations. Thus,

$$\left(m \frac{\sin \theta'}{\sin \theta}\right)^2 = 1 - \eta \text{ approximately,}$$

where

$$\eta = \frac{2 \frac{d^2}{m} \left(\mathrm{M} \frac{\partial}{\partial \mathrm{M}} + \mathrm{N} \frac{\partial}{\partial \mathrm{N}}\right) \Phi}{\rho^2 \mathrm{L}'^2 + 8\Phi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now from the definitions of α, β, γ (Part I)

$$\frac{\partial}{\partial \mathrm{M}} = \frac{\partial \alpha}{\partial \mathrm{M}} \frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial \mathrm{M}} \frac{\partial}{\partial \beta} + \frac{\partial \gamma}{\partial \mathrm{M}} \frac{\partial}{\partial \gamma}$$

$$= 2 \frac{\mathrm{M} - s\mathrm{M}'}{(s - m)^2} \frac{\partial}{\partial \alpha} + \frac{\mathrm{M} - s\mathrm{M}}{(s - m)^2} \frac{\partial}{\partial \beta}, \text{ since } \mathrm{M} - m\mathrm{M}' = 0 \text{ here,}$$

$$= \frac{\mathrm{M} - s\mathrm{M}'}{(s - m)^2} \left(2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right),$$

and similarly

$$\frac{\partial}{\partial \mathrm{N}} = \frac{\mathrm{N} - s\mathrm{N}'}{(s - m)^2} \left(2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right),$$

so that

$$\left(\mathrm{M} \frac{\partial}{\partial \mathrm{M}} + \mathrm{N} \frac{\partial}{\partial \mathrm{N}}\right) \Phi = \frac{\mathrm{M}\,(\mathrm{M} - s\mathrm{M}') + \mathrm{N}\,(\mathrm{N} - s\mathrm{N}')}{(s - m)^2} \left(2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right) \Phi$$

$$= \frac{\mathrm{M}\rho \cos \phi + \mathrm{N}\rho \sin \phi}{d^2} \left(2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right) \Phi$$

$$= - \frac{m\rho^2}{d^3} \left(2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right) \Phi, \text{ from (A).}$$

If coma only be present $\Phi = - \frac{\sigma_2}{2} \alpha\beta$; so that, from (2),

$$\eta\,(\rho^2 \mathrm{L}'^2 + 8\Phi d) = \frac{2d^2}{m} \frac{m\rho^2}{d^3} \frac{\sigma_2}{2}\,(2\beta + \alpha),$$

$$i.e., \quad \eta d \rho^2 \mathrm{L}'^2 = \rho^2 \sigma_2 \alpha,$$

since β, Φd, are each of order four in small quantities,

$$i.e., \ \eta d^3 = \sigma_2 \rho^2.$$

But in the geometrical coma figure (§ 14), if Y_1 be the distance of G from the axis of the instrument,

$$\mathrm{GQ} = \frac{\sigma_2}{2} \frac{\rho^2 \mathrm{Y}_1}{d^3},$$

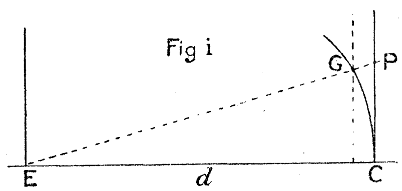
so that

$$\eta \mathrm{Y}_1 = 2\mathrm{GA}.$$

Thus
$$\frac{\sin\theta}{\sin\theta'} = m(1+\varepsilon) \quad \text{where} \quad \varepsilon Y_1 = GA,$$

and this is the result used in § 20.

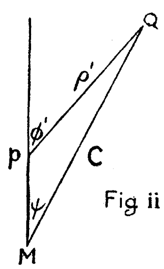
§ 22. *Curvature of the Field and Astigmatism.*—The co-efficients σ_3, σ_4 govern curvature of the field and astigmatism; if $\sigma_4 = 0$ astigmatism is absent. The appropriate forms of the general integral are considered in the Appendix.



Let $\sigma_4 = 0$ and let the diagram (fig. i) represent curvature of the field — the notation being as in Part I; P is the Gaussian image and CP $= Y_1$. Then GP $= \frac{\sigma_3 Y_1^2}{2d^2}$, to the first order, where EP cuts the image surface in G so that G is the actual focus corresponding to P, the Gaussian focus. Thus if in Appendix § 8 (4) we put $X = \frac{\sigma_3 Y_1^2}{2d^2} = GP$, *i.e.*, if we consider a receiving plane passing through G we have the integral,

$$2\pi \int_0^{\rho_1} \rho d\rho J_0\left(\frac{\kappa\rho C}{d}\right) \quad \text{leading to} \quad 2\,\frac{J_1\left(\frac{\kappa\rho_1 C}{d}\right)}{\left(\frac{\kappa\rho_1 C}{d}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this gives the Airy disc, the centre of which is at the point $C = 0$, *i.e.*, from Appendix § 8 (3), a point at a distance $\frac{XY_1}{d}$ below the Gaussian image and in the plane X.



Let Q be the point C (fig. ii) in the plane $X = \frac{\sigma_3 Y_1^2}{2d^2}$ and let p be the orthogonal projection of P upon this plane and M the point from which C is measured. It is evident from the relation $Mp = \frac{XY_1}{d}$ that M is the point G of fig. i.

§ 23. Thus we have the general result; for varying positions of the receiving plane X, the diffraction pattern is a ring system whose centre lies upon the line joining the midpoint of the exit pupil E to the Gaussian image P, and for the particular plane passing through G the central intensity is unity. It is interesting to interpret these results in terms of the Petzval sum ϖ in order to discover what value of ϖ is permissible. It may be remarked first, however, that if one object point only be present (a star in the case of a telescopic system) then a suitable change of focus will give the Airy disc whatever value ϖ may assume; but if several stars be in the field of view, to be examined simultaneously, then unlimited change of focus is not allowable. For one star disc will satisfy the Rayleigh limit for one set of positions of the plane X, another for another set, and so on, and that value of ϖ is required which will bring within the Rayleigh limit all the star-discs for at least one position of the plane X.

Let $\frac{\kappa X' \rho_1^2}{2d^2} = 2\pi n$, where X′ measures the lateral shift of the receiving plane from its position when passing through G (fig. 1). Then the intensity along the line EP is given by

$$\left(\frac{\sin \pi n}{\pi n}\right)^2, \qquad (1)$$

as appeared when considering ordinary out-of-focus effects. And X′ = X + GP if X, X′ be measured in the same direction; also $\sigma_3 = \varpi d^2$, since $\sigma_3 - \sigma_4 = \varpi d^2$ in general, where ϖ is the Petzval sum. Thus

$$2\text{GP} = \varpi Y_1^2 \qquad (2)$$

Now it is known that $n = \frac{1}{4}$ brings us to the Rayleigh limit (§11, Pt. II), *i.e.*, in this case the value of the intensity is ·80; so that P will satisfy the limit, provided that

$$\frac{\kappa\rho^2}{2d^2}\frac{\varpi Y_1^2}{2} \not> \frac{\pi}{2},$$

i.e.,

$$\varpi Y_1^2 \not> \left(\frac{d}{\rho}\right)^2 \frac{2\pi}{\kappa};$$

or, since $\kappa\lambda = 2\pi$, where λ is the wave-length of the light considered, provided that

$$\varpi Y_1^2 \not> \left(\frac{d}{\rho}\right)^2 \lambda. \qquad (3)$$

If, now, there be several stars whose images are ranged from P to G we may place the receiving plane midway between G and P; this is equivalent to the admission of a value of GP—and therefore of ϖ—twice as large as before. Thus the value of ϖ as determined (as a limit) by (3) may be doubled, or, if ϖ be kept fixed, the radius of the field may be multiplied by $\sqrt{2}$.

As a numerical example, let us consider the case of a telescope in which

$$d = 30\rho,$$

i.e., the ratio of the aperture to twice the focal length is 1 : 30. Then (3) gives

$$\varpi Y_1^2 \not> 900\lambda,$$

i.e., $\varpi Y_1^2 \not> 0{\cdot}54$, taking $\lambda = 0{\cdot}0006$ mm.

Thus, taking $Y_1 = 2{\cdot}5$ cm., the limit for ϖ is

$$\varpi = 0{\cdot}0009 \text{ (approx.)},$$

and to find the limit when change of focus is admitted this figure must be doubled.

§24. In the general case, if $\sigma_4 \neq 0$, astigmatism also is present and the general expression for the light intensity is obtained in Appendix § 8. The resulting integrals admit of general integration and then can be made, immediately, specialisations of considerable importance. Before proceeding, however, to this it may be well to draw some general conclusions from the form of the integrals :—

(1) the integrals are unaffected by a change in the sign of ϕ' (*i.e.*, by a change in the sign of ψ) : thus for any position of the receiving plane X the light contours are symmetrical about the line $\phi' = 0$:

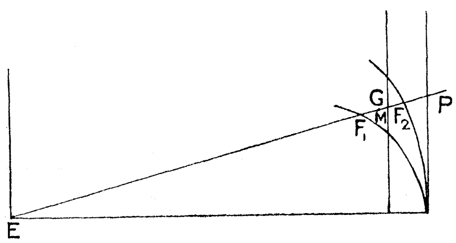
(2) the integrals are unaffected by the addition of π to ψ : the origin, therefore, is a centre of symmetry for the contours :

(3) on differentiation with respect to ψ it becomes evident, by putting $\psi = 0$, that the contours cut the line $\psi = 0$ at right-angles.

These results might perhaps have been expected from the geometrical theory of astigmatism.

From fig. ii, § 22, it is evident that (ξ, ψ) have replaced (z, ϕ') for the general case of a plane near to the Gaussian image plane ; ξ is the distance of the pole Q from the origin M measured in the usual co-ordinates, and M is the centre of the diffraction pattern for the particular plane under consideration.

§ 25. *The Distribution of Intensity along the Central Line.*—Let P be the Gaussian image point and E the axial point of the exit pupil ; join EP. Then, as in the case of simple curvature of the field, the central maxima of the diffraction patterns for various positions of the receiving plane will all lie upon the line EP, and this line may be called the central line. Let a receiving plane cut the central line in G ; let F_1, F_2 be the 'primary' and 'secondary' foci, and M the mid-point of the astigmatic separation $F_1 F_2$.



It is important to obtain a knowledge of the distribution of intensity along the central line EP. This may be obtained from the general solutions of the integrals given in the Appendix ; or, it may be obtained more simply as follows.

From Part I the distances of F_1, F_2 from the Gaussian image plane are given by $2Xd^2 = Y_1^2(\sigma_3 + 2\sigma_4)$, $2Xd^2 = Y_1^2\sigma_3$ respectively, where Y_1 is the distance of P from the axis of the instrument. Thus $[X]d^2 = Y_1^2\sigma_4$ gives the astigmatic separation, *i.e.*, $F_1 F_2$ approximately (whence it is evident that the coefficient σ_4 controls the astigmatism).

Now the general integral to be considered is

$$\mathscr{I} = \int_0^{\rho_1} \rho d\rho e^{i\gamma\rho^2} \int_0^{2\pi} e^{i\beta \sin^2\phi} \, d\phi, \ldots\ldots \text{App. } \S 9\ (3)$$

since $\xi = 0$ upon EP ; where $\gamma\rho^2 = \frac{\kappa\rho^2}{2d^2}\left\{X - \frac{Y_1^2}{2d^2}(\sigma_3 + 2\sigma_4)\right\}$, $\beta = \frac{\kappa\rho^2}{2d^2}\frac{Y_1^2}{d^2}\sigma_4$.

Write
$$\gamma\rho^2 = \frac{\kappa\rho^2}{2d^2}\left\{X - \frac{Y_1^2}{2d^2}(\sigma_3 + \sigma_4)\right\} - \frac{\kappa\rho^2}{4d^4}\sigma_4 Y_1^2$$
$$= \Gamma\rho^2 - \frac{\beta}{2},$$
where $\Gamma = \frac{\kappa}{2d^2}\left\{X - \frac{Y_1^2}{2d^2}(\sigma_3 + \sigma_4)\right\}$, and measures, therefore, the separation of G from the point M.

Then
$$\mathscr{I} = \int_0^{\rho_1} \rho d\rho e^{i\Gamma\rho^2} \int_0^{2\pi} e^{i\beta \sin^2\phi - i(\beta/2)}\, d\phi$$
$$= \int_0^{\rho_1} \rho d\rho e^{i\Gamma\rho^2} \int_0^{2\pi} e^{-i(\beta/2)\cos 2\phi}\, d\phi$$
$$= 2\pi \int_0^{\rho_1} \rho d\rho e^{i\Gamma\rho^2} J_0(\beta/2) \quad . \quad . \quad . \quad . \quad (1)$$

Then if we write $\beta = 2\alpha\rho^2$,
$$\mathscr{I} = 2\pi \int_0^{\rho_1} \rho d\rho e^{i\Gamma\rho^2} J_0(\alpha\rho^2)$$
$$= \pi \int_0^{\rho_1^2} e^{i\Gamma u} J_0(\alpha u)\, du, \quad . \quad . \quad . \quad . \quad (2)$$

by change of variable. The intensity at the image point when $\sigma_3 = \sigma_4 = 0$ must be unity, *i.e.*, when $\Gamma = 0$, $\alpha = 0$; then (2) reduces to $\pi\rho_1^2$, so that the expression whose squared modulus is equal to the intensity is
$$\frac{1}{\rho_1^2}\int_0^{\rho_1^2} e^{i\Gamma u} J_0(\alpha u)\, du \quad . \quad . \quad . \quad . \quad (3)$$

Let now n be the number of wave-length units of astigmatism, *i.e.*, let $2\pi n$ be the excess of phase difference for two extreme rays from the exit pupil to F_1 over that for two extreme rays to F_2; then
$$2\alpha\rho_1^2 = \frac{\kappa\rho_1^2}{2d^2}[X] = 2\pi n = 2\mu \text{ (say)}.$$

Similarly, if n_1 measure the distance MG,
$$\Gamma\rho_1^2 = 2\pi n_1 = \lambda \text{ (say)};$$
then (3) becomes
$$\sqrt{I} = \left|\int_0^1 e^{i\lambda t} J_0(\mu t)\, dt\right| \quad . \quad . \quad . \quad . \quad (4)$$

It is evident that the intensity distribution depends upon the *difference* between the curvatures of the two focal surfaces.

§26. *Some Special Cases.*—(1) Let $\mu = 0$, *i.e.*, consider a system free from astigmatism, but not necessarily free from curvature of the field. Then (4) reduces to
$$\int_0^1 e^{i\lambda t}\, dt, \quad . \quad . \quad . \quad . \quad (1)$$

so that the intensity is given by $\left(\frac{\sin \pi n_1}{\pi n_1}\right)^2$ as in § 23 (1), and the field may be curved or not.

(2) Let $\lambda = 0$, *i.e.*, consider the intensity at M, the mid-point of the astigmatic separation. Then (4) reduces to

$$\int_0^1 J_0(\mu t)\, dt. \qquad (2)$$

(3) Let $\lambda - \mu = 0$, *i.e.*, consider the intensity at the primary focal point; then (1) reduces to

$$\int_0^1 e^{i\mu t} J_0(\mu t)\, dt, \qquad (3)$$

while at the secondary focal point $\lambda + \mu = 0$, so that we have

$$\int_0^1 e^{-i\mu t} J_0(\mu t)\, dt, \qquad (4)$$

and it is evident that the intensities at the focal points are equal; indeed, since in (4) a change in the sign of λ leaves the modulus of the integral unaltered, it follows that the intensities are symmetrical about the point M.

Numerical Results.—From (2) above the intensity at the mid-point M is given by

$$\sqrt{I} = \int_0^1 J_0(\mu t)\, dt,$$

and $\int_0^1 J_0(1 \cdot 2t)\, dt = 0 \cdot 8863$, so that $I = 0 \cdot 7855$. The quantity of astigmatism represented, therefore, by $\mu = 1 \cdot 2$ brings us to the Rayleigh limit: let [X] be the astigmatic separation; then $[X] = \frac{2d^2}{\kappa \rho^2} 2\mu$. As a typical case, let $d = 30\rho$ (in a telescopic system), and let $\lambda = 0 \cdot 0006$ mm.; then

$$[X] = \frac{2\lambda}{2\pi}(30)^2 \, . \, 2 \cdot 4 = 0 \cdot 41 \text{ mm. (approx.)}.$$

This then gives the maximum permissible amount of astigmatism.

By the evaluation of (3), (4) it may be shown that the intensity at either of the principal foci is given by the squared modulus of the expression

$$\sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} \left(\frac{i\mu}{2}\right)^n \frac{1}{\lfloor n+1}, \qquad (5)$$

and this lends itself to rapid numerical calculation. The intensity in any other case is most easily obtained, perhaps, from quadratures of § 25 (4); it is given also, however, by the squared modulus of the expression

$$\frac{1}{i(\mu - \lambda)} \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} \left\{\frac{\mu}{2(\mu - \lambda)}\right\}^n \sum_{p=1}^{\infty} \frac{\{i(\mu - \lambda)\}^{n+p}}{\lfloor n+p}, \qquad (6)$$

and this, in the cases of practical importance, lends itself to fairly rapid numerical evaluation.

In § 29 below it is shown that astigmatism of amount $\mu = 1$ leads to an intensity represented by 0·780 at either principal focus: on account of the shape of the contours (Fig. 4) an image plane will be placed naturally so as to pass through a principal focus, for in this position the resolving power will in general be greater than in any other position. From this point of view, therefore, the maximum permissible astigmatism is about 0·34 mm.

§ 27. *The Light Contours.*—The general integral to be considered is

$$\vartheta(\Gamma, \psi) = \int_0^{\rho_1} \rho d\rho \int_0^{2\pi} e^{i\Gamma\rho^2 - i\zeta \cos \overline{\phi - \psi} - i(\beta/2)\cos 2\phi} \, d\phi$$

$$= \int_0^{\rho_1} \rho d\rho \int_0^{2\pi} e^{i\Gamma\rho^2 + i\zeta \sin \overline{\phi - \psi} + i(\beta/2)\cos 2\phi} \, d\phi,$$

by means of the substitution of $\left(\frac{\pi}{2} + \phi\right)$ for ϕ.

Now
$$\vartheta\left(-\Gamma, \psi + \frac{\pi}{2}\right) = \int_0^{\rho_1} \rho d\rho \int_0^{2\pi} e^{-i\Gamma\rho^2 - i\zeta \sin \overline{\phi - \psi} - i(\beta/2)\cos 2\phi} \, d\phi,$$

so that
$$|\vartheta(\Gamma, \psi)| = \left|\vartheta\left(-\Gamma, \psi + \frac{\pi}{2}\right)\right|.$$

Thus, a change in the sign of Γ produces the same effect as an addition of $\frac{\pi}{2}$ to ψ; the diffraction patterns at equal distances on either side of $\Gamma = 0$, *i.e.*, of the point M, are therefore identical; the one is turned, however, relative to the other through an angle $\frac{\pi}{2}$. It is interesting to observe that the equations of the light contours may be written as follows in particular cases:—

(1) those for either of the planes through the foci F_1, F_2,

$$\left|\int_0^{\rho_1} \rho d\rho \int_0^{2\pi} e^{-i\zeta \cos(\phi - \psi) + 2i\mu \cos^2 \phi} \, d\phi \right| = \text{const.};$$

(2) those for the plane through the mid-point M of the astigmatic separation,

$$\left|\int_0^{\rho_1} \rho d\rho \int_0^{2\pi} e^{-i\zeta \cos(\phi - \psi) - i\mu \cos 2\phi} \, d\phi \right| = \text{const.}$$

§ 28. From Appendix § 10 the light intensity in the general case is given by the squared modulus of

$$2 \sum_{n=0}^{\infty} \frac{y_{2n}}{2^n \lfloor n} \frac{U_{n+1}}{Z^{2n+2}}, \qquad (1)$$

where
$$U_{n+1} = \int_0^z Z^{n+1} e^{i\nu Z^2} J_n(Z) \, dZ$$

$$= \frac{e^{i\nu Z^2} Z^{2n+2}}{(2i\nu Z^2)^{n+1}} V_{n+1},$$

and
$$V_{n+1} = (2i\nu Z^2)^{n+1} \frac{J_{n+1}(Z)}{Z^{n+1}} - (2i\nu Z^2)^{n+2} \frac{J_{n+2}(Z)}{Z^{n+2}} + \dots, \qquad (2)$$

so that V_n is a generalised Lommel Function. The expression whose squared modulus gives the light intensity may therefore be written

$$2 \sum_{n=0}^{\infty} \frac{y_{2n}}{2^n \lfloor n} \frac{V_{n+1}}{(2\iota\nu Z^2)^{n+1}}. \qquad (3)$$

It will be remembered that the term νZ^2 measures the separation of the image plane from the primary focus.

In the cases of practical importance (3) reduces very considerably. For example, if we consider the plane passing through the primary focus, we have $2Xd^2 = Y_1^2(\sigma_3 + 2\sigma_4)$; so that in the notation above used $\nu Z^2 = 0$, and then

$$\frac{V_{n+1}}{(2i\nu Z^2)^{n+1}} = \frac{J_{n+1}(Z)}{Z^{n+1}}.$$

The expression (3) becomes, therefore,

$$2 \sum_{n=0}^{\infty} \frac{y_{2n}}{2^n \lfloor n} \frac{J_{n+1}(Z)}{Z^{n+1}}; \qquad (4)$$

and this will give the intensity over the plane passing through the primary focus; by the symmetry already shown to exist it will give also the intensity over the plane passing through the secondary focus. We must remember, however, that the light contours are turned now through a right angle.

For any given plane much information can be obtained from a consideration of the intensity distribution along the principal-axes of the contours for that plane, *i.e.*, along the lines defined by $\psi = 0$, $\psi = \frac{\pi}{2}$. If $\psi = 0$, then

$$y_{2n} = \frac{\lfloor 2n}{2^n \lfloor n} b^n = \frac{\lfloor 2n}{2^n \lfloor n} (4\iota\mu)^n,$$

and (3) takes the form

$$2 \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} (i\mu)^n \frac{V_{n+1}}{(2i\nu Z^2)^{n+1}}. \qquad (5)$$

From the symmetry already indicated this expression will give the intensity not only along the line $\psi = 0$, but also along the line $\psi = \frac{\pi}{2}$. For the primary plane $\nu = 0$ and then (5) becomes

$$2 \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} (i\mu)^n \frac{J_{n+1}(Z)}{Z^{n+1}}, \qquad (6)$$

while the intensity at the primary focus is obtained by writing Z = 0, and is given by

$$2 \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} (i\mu)^n \frac{1}{2^{n+1} \lfloor n+1}, \qquad (7)$$

or

$$\sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2 \lfloor n+1} \left(\frac{i\mu}{2}\right)^n. \qquad (8)$$

These expressions are all susceptible of rapid numerical calculation.

The plane passing through the secondary focus is defined by $\nu Z^2 = -2\mu$, and that passing through the mid-point of the astigmatic separation by $\nu Z^2 = -\mu$. The intensity distribution along the line $\psi = 0$ in the secondary plane is given therefore by (5), which takes the form

$$\frac{1}{2i\mu} \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} \frac{V_{n+1}}{4^n}. \qquad (9)$$

where

$$V_{n+1} = \left(\frac{4i\mu}{Z}\right)^{n+1} J_{n+1}(Z) + \left(\frac{4i\mu}{Z}\right)^{n+2} J_{n+2}(Z) + \dots .$$

The intensity along either of the axes of the contours in the mid-astigmatic plane is given by (5), which takes here the form

$$\frac{1}{i\mu} \sum_{n=0}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} \frac{V_{n+1}}{2^n} \qquad (10)$$

where

$$V_{n+1} = \left(\frac{2i\mu}{Z}\right)^{n+1} J_{n+1}(Z) + \left(\frac{2i\mu}{Z}\right)^{n+2} J_{n+2}(Z) + \dots .$$

In all these formulæ Z is the distance from the centre (of the contours) to the point under consideration; usually Z is multiplied by the factor cos ψ, but here $\psi = 0$.

It has been pointed out that owing to the symmetry of § 27, the above formulæ are sufficient to give the intensity distribution along each of the axes of the contours, and the formulæ are convenient for numerical evaluation. Other formulæ may be obtained, but they are not so convenient for numerical work: *e.g.*, if $\psi = \frac{\pi}{2}$ then $Z = 0$ since Z contains the factor cos ψ. Thus (3) becomes

$$2 \sum_{n=0}^{\infty} \frac{y_{2n}}{2^n \lfloor n} \frac{V_{n+1}}{(2i\nu Z^2)^{n+1}},$$

where now

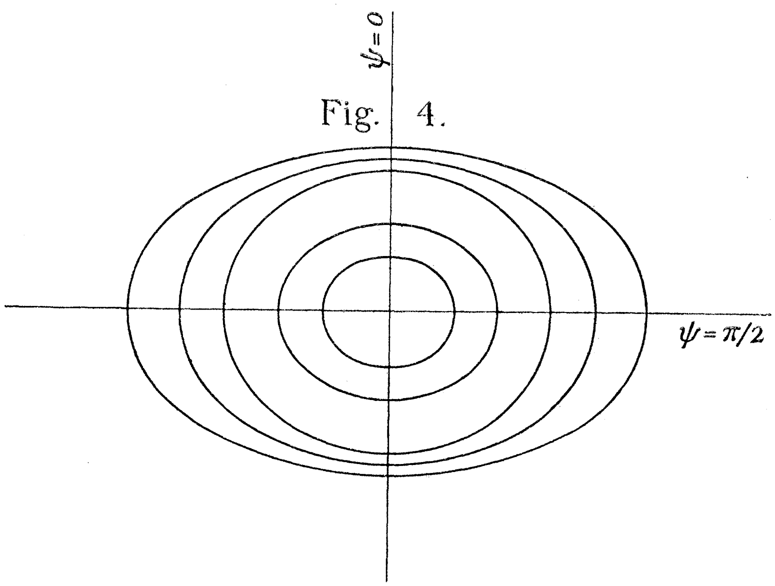
$$V_{n+1} = \frac{(i\nu Z^2)^{n+1}}{\lfloor n+1} - \frac{(i\nu Z^2)^{n+2}}{\lfloor n+2} + \dots ;$$

νZ^2 of course will not vanish because ν contains a factor $(\cos \psi)^{-1}$. In the particular case of the primary plane (3) becomes

$$\sum_0^\infty \frac{y_{2n}}{2^{2n} \lfloor n \lfloor n+1}. \qquad (11)$$

The general results now obtained have been derived from equation Appendix § 9 (3); they might equally well have been derived from equation (4) of that paragraph.

Fig. 4.



Light Contours upon plane passing through Primary Focus, in the presence of Astigmatism of amount $\mu = 1$.

§ 29. *Numerical Results.*—By means of the above formulæ the following results have been obtained for the case in which $\mu = 1$; ζ measures the distance from the centre of the diffraction pattern:—

	0	1	2	3	4	5
$\zeta =$	0	1	2	3	4	5
$\psi = 0 : I =$	0·780	0·600	0·327	0·053	0·004	0·019
$\psi = \frac{\pi}{2} : I =$	0·780	0·654	0·337	0·143	0·087	0·056

These figures show the existence of focal lines as modified by diffraction; the contours are as in the diagram (Fig. 4). The above figures refer to the plane through the primary focus, but as indicated above the contours for the plane through the secondary focus are exactly similar except for a rotation as a whole through a right angle.

It will be seen that the above amount of astigmatism, denoted by $\mu = 1$, brings us very approximately to the Rayleigh limit for the planes passing through the two foci: and in this case the least radius of the dark 'ring'—given by $\psi = 0$ above—is a little greater than 4 as compared with 3·8 in the AIRY disc. Thus, for the purposes of star resolution this amount of astigmatism is negligible. For the mid-astigmatic plane the radius of the first dark ring is greater than that for the 'focus' planes; indeed, it is intermediate between the cases $\psi = 0$ and $\psi = \pi/2$ shown above. Thus, for star resolution the two extreme planes are used instead of the middle plane.

§ 30. *Effect of Distortion.*—Let all the aberration coefficients vanish except σ_5, *i.e.*, let distortion be the only aberration present. Then (*cf.* Appendix) the light intensity I is given by

$$\sqrt{\mathrm{I}} = 2\,\frac{\mathrm{J}_1\left(\dfrac{\kappa\rho\mathrm{C}}{d}\right)}{\dfrac{\kappa\rho\mathrm{C}}{d}}, \quad \ldots \ldots \quad (1)$$

where

$$\mathrm{C}^2 = \left(\frac{\sigma_5}{2d^3}\mathrm{Y}_1^{\,3}\right)^2 - 2\rho' \cos\phi' \left(\frac{\sigma_5}{2d^3}\mathrm{Y}_1^{\,3}\right) + \rho'^2, \quad \ldots \ldots \quad (2)$$

and we consider only the Gaussian image plane given by $\mathrm{X} = 0$. Let P be the pole (ρ', ϕ'), G the Gaussian image point and A the geometrical image point, as displaced by distortion. Then, from Part I

$$\mathrm{GA} = \frac{\sigma_5 \mathrm{Y}_1^{\,3}}{2d^3},$$

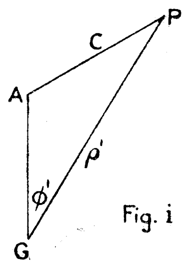


Fig. i

and $\mathrm{C}^2 = \mathrm{AG}^2 - 2\,\mathrm{GP}\,.\,\mathrm{GA}\cos\phi' + \mathrm{GP}^2 = \mathrm{AP}^2$ from (2), *i.e.*, $\mathrm{C} = \mathrm{AP}$: also external angle $\mathrm{GAP} = \psi$ (Appendix § 11).

We have, therefore, as the diffraction pattern, the AIRY disc with centre A. Thus, the pattern is the same as in the non-distortion case except that it has a new centre at the new (geometrical) image point.

Let now $\mathrm{X} \neq 0$, *i.e.*, consider the effect in a plane near to the Gaussian image plane; here now

$$\mathrm{U}_1 + r_0 = \frac{\mathrm{X}}{2}\left(\frac{\rho}{d}\right)^2 + \mathrm{C}'\,\frac{\rho}{d}\cos(\phi - \psi'), \quad \ldots \ldots \quad (3)$$

where

$$\mathrm{C}'^2 = \left(\frac{\sigma_5}{2d^3}\mathrm{Y}_1^{\,3} - \mathrm{X}\,\frac{\mathrm{Y}_1}{d}\right)^2 - 2\rho'\left(\frac{\sigma_5 \mathrm{Y}_1^{\,3}}{2d^3} - \mathrm{X}\,\frac{\mathrm{Y}_1}{d}\right)\cos\phi + \rho'^2 \quad (\textit{cf. } \text{Appendix § 11.})$$

and ψ' is independent of ρ, ϕ. Thus (*cf.* Appendix) we are led to the integral

$$\int_0^{\rho_1} \rho\, d\rho\, e^{\frac{i\kappa\rho^2 \mathrm{X}}{2d^2}}\, \mathrm{J}_0\left(\frac{\kappa\rho\mathrm{C}'}{d}\right) \quad \ldots \ldots \quad (4)$$

and this is the integral which occurred in the consideration of ordinary out-of-focus effects in the absence of aberration; it leads to ordinary Lommel functions and the results obtained before may, therefore, be quoted.

Let fig. ii be the orthogonal projection upon the plane X of fig. 1; let Q be a point upon GA such that

$$GA = X \frac{Y_1}{d} \quad \ldots \ldots \ldots \ldots \quad (5)$$

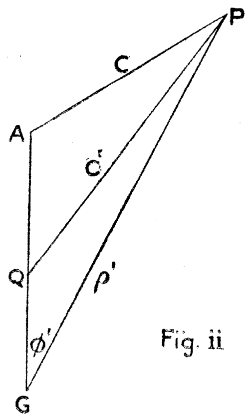


Fig. ii

so that $C' = QP$.

The diffraction pattern, therefore, for a given value of X consists of a series of concentric circles with Q as centre; from (5) above the locus of Q for varying values of X is the straight line joining the distortion image point and the centre of the exit pupil.

From the Appendix the expression whose squared modulus gives the intensity is

$$\frac{e^{2\pi in}}{2\pi in}\left\{(4\pi in)\frac{J_1(z)}{z} - (4\pi in)^2\frac{J_2(z)}{z^2} + \ldots\right\} \quad \ldots \ldots \quad (6)$$

where $\kappa X \rho^2 = 4\pi nd^2$, $\kappa\rho C' = zd$.

The diffraction pattern is the same as that given by out-of-focus conditions in the absence of aberrations, except that there is a change in the position of the centre of the rings corresponding to a change in the focussing plane.

PART III.

§ 1. *The Annular Aperture.*—Let the central portion of the exit pupil be blocked out so that the aperture consists of an annulus between circles of radii ρ_1, $\alpha\rho_1$ where $0 \leq \alpha \leq 1$; then in the notation of Part II., § 1

$$S = \pi\rho_1^2(1 - \alpha^2)$$

and the expression to be considered is

$$\frac{1}{\pi\rho_1^2(1-\alpha^2)}\int_{\alpha\rho_1}^{\rho_1}\rho d\rho\int_0^{2\pi}e^{i\kappa(U_1+r_0)}d\phi \quad \ldots \ldots \ldots \quad (1)$$

Neglecting the aberrations of the system, so that

$$U_1 + r_0 = -\frac{\rho\rho'}{d}\cos(\phi - \phi'),$$

in the usual notation, the expression (1) becomes

$$\frac{1}{\pi\rho_1^2(1-\alpha^2)}\int_{\alpha\rho_1}^{\rho_1}\rho d\rho\int_0^{2\pi}e^{-i\kappa(\rho\rho'/d)\cos(\phi-\phi')} = \frac{2}{\rho_1^2(1-\alpha^2)}\int_{\alpha\rho_1}^{\rho_1}\rho\, d\rho\, J_0\left(\frac{\kappa\rho\rho'}{d}\right)$$

$$= \frac{2}{1-\alpha^2}\left\{\frac{J_1(z)}{z} - \alpha^2\frac{J_1(\alpha z)}{\alpha z}\right\}, \quad \ldots \quad (2)$$

where $zd = \kappa\rho_1\rho'$ as usual. Thus the light-intensity I is given by

$$\sqrt{\mathrm{I}} = \frac{2}{1-\alpha^2}\left\{\frac{\mathrm{J}_1(z)}{z} - \alpha^2\frac{\mathrm{J}_1(\alpha z)}{\alpha z}\right\}. \quad . \quad . \quad . \quad . \quad (3)$$

This clearly reduces to the Airy disc when $\alpha = 0$, and in any case the diffraction pattern consists of a series of concentric circles, the centre being at the Gaussian image point. The radii of the dark rings are given by the roots of the equation in z

$$\mathrm{J}_1(z) - \alpha\,\mathrm{J}_1(\alpha z) = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

while the maxima are given by the roots of

$$\mathrm{J}_2(z) - \alpha^2\,\mathrm{J}_2(z) = 0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The first root of (4) in the case $\alpha = \frac{1}{2}$ is just less than $3\cdot15$; with full aperture $\alpha = 0$ the first root is $3\cdot83$. As α tends to unity, the expression (3) tends to $\mathrm{J}_0(z)$ and the first root of this is $2\cdot40$; thus an increase in α gives a decrease in the radius of the first dark ring. The first root of (5) is $4\cdot8$ approximately if $\alpha = \frac{1}{2}$, while when $\alpha = 1$ it is $3\cdot83$; and these compare with the value $5\cdot14$ when $\alpha = 0$. There is, therefore, a decided gain in resolving power, since this is measured by the radius of the first dark ring; and the radii of the first bright rings decrease as α increases. This gain is, of course, counter-balanced by a loss in light due to the cutting down of the exit pupil, and thus the advantage obtainable depends upon the brightness of the stars under observation. The diagram given (fig. 5) shows the intensity distribution; it is evident that the decrease in α makes the successive rings relatively brighter while decreasing their size.

§ 2. Let us assume now that first-order spherical aberration is present combined with out-of-focus effects. Then, as shown in Part II., the expression to be considered for full aperture effects ($\alpha = 0$) is

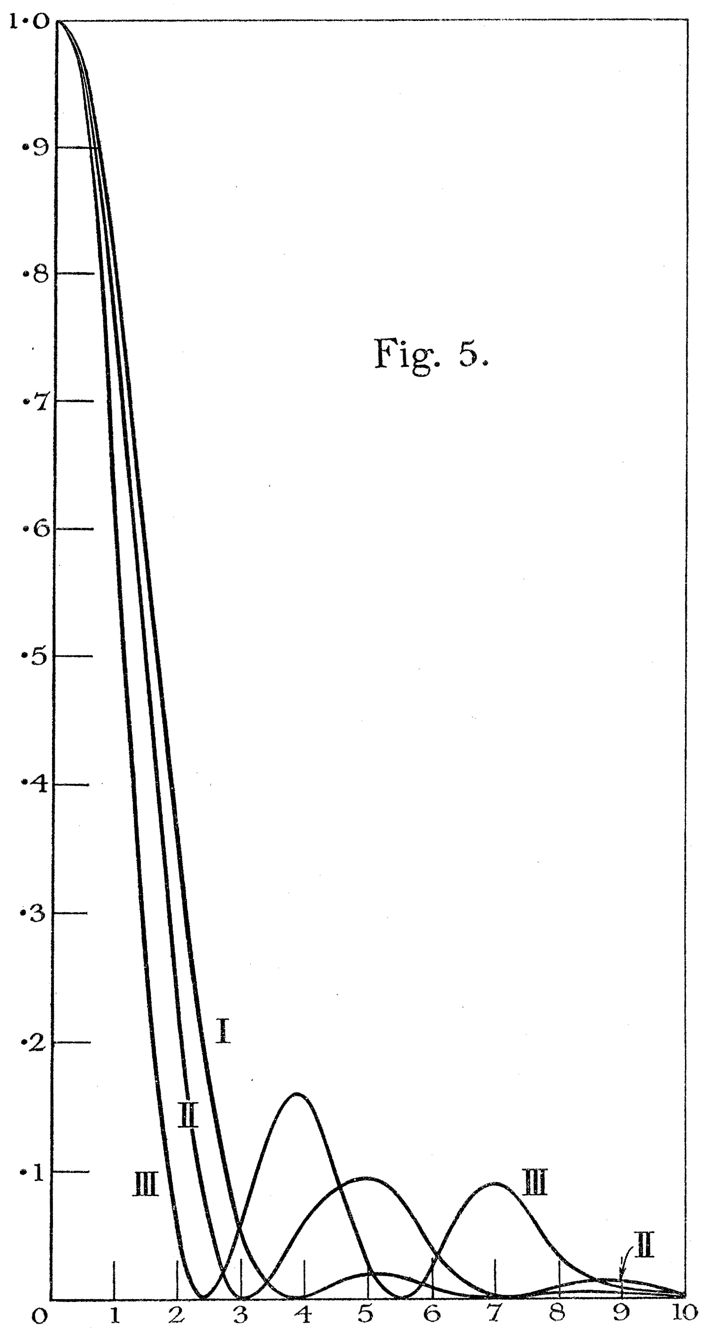
$$\int_0^1 e^{2\pi i(nt+n_1t^2)}\,\mathrm{J}_0(z\sqrt{t})\,dt, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n, n_1 have their usual meanings. From this has to be subtracted the integral due to the portion of the exit pupil stopped out, in order to obtain the effect due to the annular aperture: the intensity at distance z from the axis of the system (assuming an object point upon the axis; there is no loss in generality in this) is given, therefore, by the squared modulus of the expression

$$\frac{1}{1-\alpha^2}\left\{\int_0^1 e^{2\pi i(nt+n_1t^2)}\,\mathrm{J}_0(z\sqrt{t})\,dt - \alpha^2\int_0^1 e^{2\pi i(\alpha^2nt+\alpha^4n_1t^2)}\,\mathrm{J}_0(z'\sqrt{t})\,dt\right\}, \quad . \quad . \quad . \quad (2)$$

where $z' = \alpha z$,

$$i.e., \quad \frac{1}{1-\alpha^2}\int_{\alpha^2}^1 e^{2\pi i(nt+n_1t^2)}\,\mathrm{J}_0(z\sqrt{t})\,dt. \quad . \quad . \quad . \quad . \quad . \quad (3)$$



Light Intensity, upon Gaussian Image Plane, with Annular Exit Pupil.

I. AIRY disc : $\sqrt{\mathrm{I}} = \dfrac{2\mathrm{J}_1(z)}{z}$ $\alpha = 0$.

II. $\sqrt{\mathrm{I}} = \frac{8}{3}\left\{\dfrac{\mathrm{J}_1(z)}{z} - \frac{1}{4}\dfrac{\mathrm{J}_1(z/2)}{z/2}\right\}$ $\alpha = \frac{1}{2}$.

III. Limiting Case : $\sqrt{\mathrm{I}} = \mathrm{J}_0(z)$ $\alpha = 1$.

In the second integral of (2) the factor α^{2s+2} has been applied to the quantity n, because the aberration of order s depends upon the $(2s+2)$th power of the radius of

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the exit pupil. Expression (3) gives the general intensity at any point and the integral may be evaluated by the methods given already. Much information may be obtained, however, from a consideration of the axial intensities; for this purpose, put $z = 0$ and we have

$$\frac{1}{1-\alpha^2}\int_{\alpha^2}^{1} e^{2\pi i\,(nt+n_1t^2+\ldots+n_st^{s+1}\ldots)}\,dt, \quad \ldots \ldots \quad (4)$$

this being the obvious generalisation to cover spherical aberration of all orders.

1. Consider first simple out-of-focus effects; (4) becomes

$$\frac{1}{1-\alpha^2}\int_{\alpha^2}^{1} e^{2\pi int}\,dt = e^{\pi in(1+\alpha^2)} \cdot \frac{\sin \pi n(1-\alpha^2)}{\pi n(1-\alpha^2)};$$

so that the light intensity is given by

$$\sqrt{\mathrm{I}} = \frac{\sin \pi n(1-\alpha^2)}{\pi n(1-\alpha^2)}. \quad \ldots \ldots \quad (5)$$

From this formula it is evident that the distance between successive dark points upon the axis is given by $[n] = 1/(1-\alpha^2)$; this tends to infinity as α tends to unity, as, indeed, is obvious from the consideration that in the limiting case the aperture reduces to a circular rim. The expression (5) shows that the intensity distribution is symmetrical about the point given by $n = 0$.

It has been shown that, if $\alpha = 0$, $n = \frac{1}{4}$ brings us to the Rayleigh limit in the case of a full circular aperture; here, however, $n = \dfrac{1}{4(1-\alpha^2)}$ brings us to the limit, so that the permissible departure from the Gaussian plane is increased in the ratio $1 : 1-\alpha^2$.

2. Consider now out-of-focus effects combined with first order aberration; then we have the expression

$$\frac{1}{1-\alpha^2}\int_{\alpha^2}^{1} e^{2\pi i\,(nt+n_1t^2)}\,dt. \quad \ldots \ldots \quad (6)$$

Now

$$\int_{\alpha^2}^{1} e^{2\pi i\,(nt+n_1t^2)}\,dt = \frac{e^{-\frac{\pi i}{2}\frac{n^2}{n_1}}}{2\sqrt{n_1}}\int_{\frac{n}{\sqrt{n_1}}+2\alpha^2\sqrt{n_1}}^{\frac{n}{\sqrt{n_1}}+2\sqrt{n_1}} e^{\frac{i\pi t^2}{2}}\,dt$$

by change of variable: thus

$$\sqrt{\mathrm{I}} = \left|\frac{1}{2(1-\alpha^2)\sqrt{n_1}}\int_{\frac{n}{\sqrt{n_1}}+2\alpha^2\sqrt{n_1}}^{\frac{n}{\sqrt{n_1}}+2\sqrt{n_1}} e^{\frac{i\pi t^2}{2}}\,dt\right|, \quad \ldots \ldots \quad (7)$$

and the value of this may be obtained at once from tables of FRESNEL'S Integrals.

(3) In general, in the presence of aberrations of all orders, the light intensity is given by the squared modulus of (4).

Numerical Results.—I find that if in (7) we substitute $\alpha^2 = \frac{1}{2}$, $n = -1$, $n_1 = 1$ we have

$$\sqrt{\mathrm{I}} = \left| \int_0^1 e^{\frac{i\pi t^2}{2}} dt \right|,$$

so that I = 0·8003. Thus with this value of α one wave-length of first-order aberration, combined with one wave-length out-of-focus, brings us to the Rayleigh limit. It is interesting to notice that at the Gaussian image point we have in this case

$$\sqrt{\mathrm{I}} = \left| \int_1^2 e^{\frac{i\pi t^2}{2}} dt \right|,$$

so that I = 0·0940. Here, again, the beneficial effects of a slight change in focus are manifest.

§ 3. Let us assume that coma is the only aberration present; then it has been seen that the most important case to consider is that of the distribution upon the Gaussian image plane. From what has preceded we have the formula

$$\sqrt{\mathrm{I}} = \int_0^1 \mathrm{J}_0(\sqrt{z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3})\, dv, \quad \ldots \ldots \quad (1)$$

giving the intensity at the point (ρ', ϕ'). The expression, therefore, in the case of the annular aperture will be

$$\sqrt{\mathrm{I}} = \frac{1}{1-\alpha^2} \int_{\alpha^2}^1 \mathrm{J}_0(\sqrt{z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3}) \quad \ldots \ldots \quad (2)$$

in the usual notation: so that the intensity upon the axis of the coma-figure is given by

$$\sqrt{\mathrm{I}} = \frac{1}{1-\alpha^2} \int_{\alpha^2}^1 \mathrm{J}_0(\beta\sqrt{v^3} - z\sqrt{v})\, dv. \quad \ldots \ldots \ldots \quad (3)$$

The dark points upon the axis are given by the roots of the equation in z

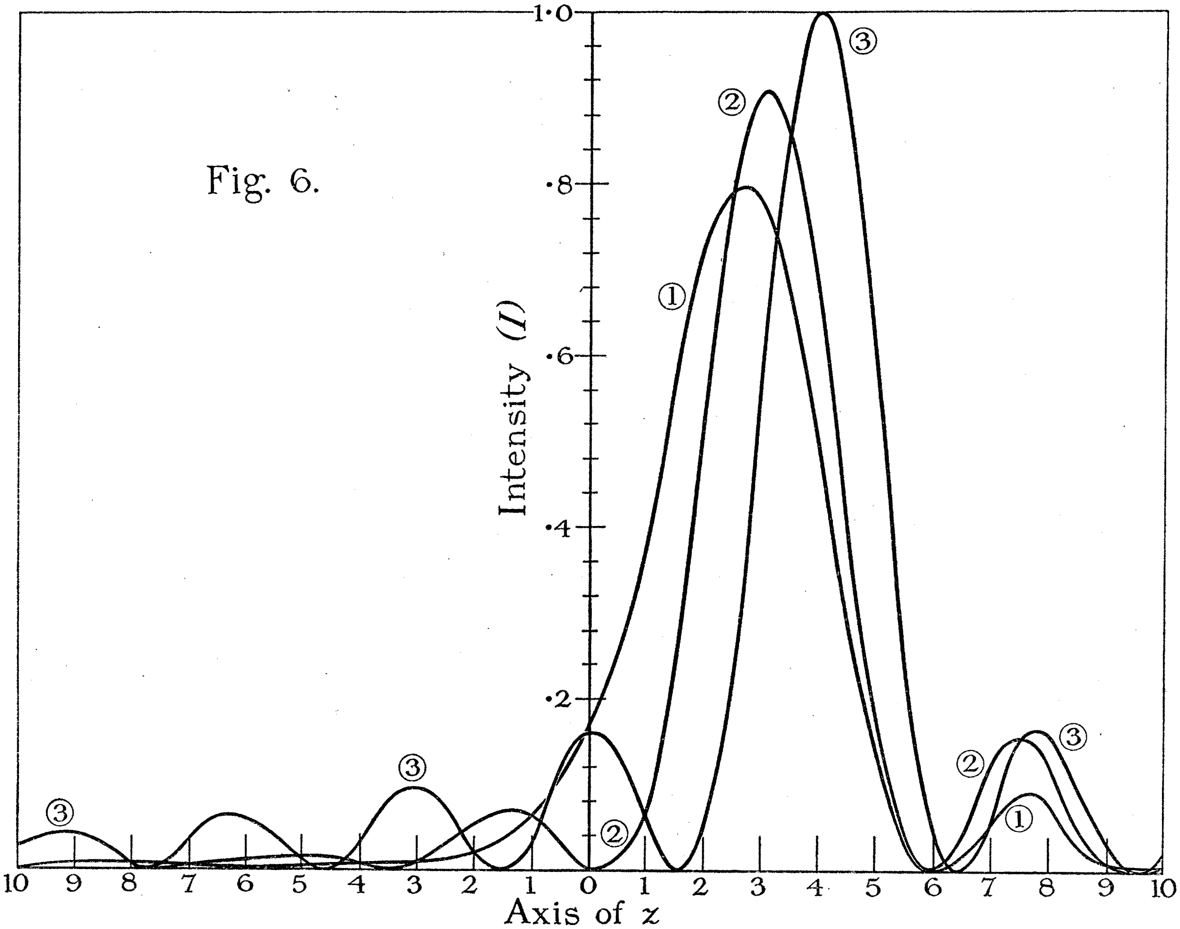
$$\int_{\alpha^2}^1 \mathrm{J}_0(\beta\sqrt{v^3} - z\sqrt{v})\, dv = 0, \quad \ldots \ldots \ldots \quad (4)$$

while the turning values of I, upon the axis, are given by the roots of the equation

$$\int_{\alpha^2}^1 \mathrm{J}_1(\beta v - z v^{1/3})\, dv = 0; \quad \ldots \ldots \ldots \ldots \quad (5)$$

as before, these will be maxima of I in general.

From the numerical evaluation of (3) it becomes evident that larger values of β are admissible, before the Rayleigh limit is reached, for the annular aperture than for the full circular aperture; with the latter it has already been shown that $\beta = 4$ reaches



Light Intensity in Presence of Annular Exit Pupil and Coma to the Amount $\beta = 4$.

$$\sqrt{I} = \frac{1}{1-\alpha^2}\int_{\alpha^2}^{1} J_0 (4\sqrt{v^3} - z\sqrt{v})\,dv.$$

(1) $\alpha = 0$, *i.e.*, full circular exit pupil.

(2) $\alpha^2 = \frac{1}{2}$, *i.e.*, annular exit pupil.

(3) $\alpha = 1$, *i.e.*, limiting case in which annulus becomes very narrow :—$\sqrt{I} = J_0 (4 - z)$.

the limit and gives a maximum intensity of 0·80 times the theoretical maximum for an aberrationless system. Putting, however, $2\alpha^2 = 1$, $\beta = 4$, $z = 3$ in (3), we have

$$\sqrt{I} = 2\int_{0\cdot 5}^{1} J_0 (4\sqrt{v^3} - 3\sqrt{v})\,dv = 0\cdot 9530,$$

so that $I = 0\cdot 9082$; larger values of β are, therefore, admissible. Indeed, putting $2\alpha^2 = 1$, $\beta = 6$, $z = 4\cdot 75$, we have

$$\sqrt{I} = 2\int_{0\cdot 5}^{1} J_0 (6\sqrt{v^3} - 4\cdot 75\sqrt{v})\,dv = 0\cdot 8712,$$

so that I $= 0 \cdot 7590$. Thus $\beta = 6$ brings us approximately to the Rayleigh limit in this case. The annular aperture leads to a smaller relative loss of intensity for a given field, or, on the other hand, it increases the field of view subject to the Rayleigh limit. This is indicated by a consideration of the limiting case, in which $\alpha \rightarrow 1$ and the aperture becomes a narrow rim; for from (3)

$$\sqrt{\mathrm{I}} = \mathrm{J}_0(\beta - z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in the limiting case. Here, of course, the absolute intensity is zero, but nevertheless for bright stars we may approach this case. Equation (6) would indicate a relative maximum of unity for all values of β, *i.e.*, for all amounts of coma present. The centre of the diffraction pattern is moved a distance β from the Gaussian image point away from the axis of the system. This equation refers only to the axis of the coma-figure. More generally we have from (2)

$$\sqrt{\mathrm{I}} = \mathrm{J}_0(u_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $u_1^2 = z^2 - 2z\beta \cos \phi' + \beta^2$ and so u_1 is the distance from the point $(\beta, 0)$ in the coma figure. The diffraction pattern is, therefore, a series of concentric circles with this point as centre, and the radius of the first dark ring is given by the first root of $\mathrm{J}_1(z) = 0$, *i.e.*, it is $2 \cdot 405$, and that of the first bright ring is $3 \cdot 832$.

There is, therefore, a decided increase in resolving power over the full aperture case in which the radius of the first dark ring is $3 \cdot 3$ approximately; the case $2\alpha^2 = 1$ is intermediate to these two.

In the diagram (fig. 6) are shown the intensity distributions along the axis of the coma-figure in the three cases $\alpha = 0, \frac{1}{2}, 1$. It will be seen that the radii of the first dark and bright rings decrease, but that the relative intensity of the bright rings is increased. It must be remembered that the relation between the absolute intensities for the different diagrams is not shown.

The diagram makes clear the change in the position of the point of maximum intensity. For not very large values of β ($\beta \not> 4$, say) the full aperture maximum occurs approximately at the point given by $3z = 2\beta$, while with the rim aperture the maximum intensity occurs at the point $z = \beta$; for other values of α the maximum point is at an intermediate position.

§ 4. The effect of curvature of the field can be deduced from the results of the preceding paragraphs, for it has been shown that, with full aperture, curvature gives ordinary out-of-focus effect combined with a change of the centre of the ring system. The equation (5), § 2, may therefore be quoted to show that a numerical value of ϖ is admissible with the annular aperture greater than that admissible with the full aperture in the ratio $1 : 1 - \alpha^2$, *i.e.*, that this larger value of ϖ will just satisfy the Rayleigh limit. Thus from Part II it is seen that a sufficiently large field is given by the limit

$$(1 - \alpha^2)\,\varpi \not> 0 \cdot 0018.$$

Curvature and astigmatism together are governed by the coefficients σ_3, σ_4; thus, in the notation of Part II, the integral whose squared modulus gives the light intensity at the point ζ, ψ upon the plane Γ is

$$\frac{1}{\pi\rho_1^2(1-\alpha^2)}\int_{\alpha\rho_1}^{\rho_1}\rho d\rho\, e^{i\Gamma\rho^2}\int_0^{2\pi}e^{-i\zeta\cos(\phi-\psi)-i(\beta/2)\cos 2\phi}\,d\phi\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the annular aperture. Consider first the intensity distribution along the central line, given by $\zeta = 0$; then the above integral reduces to

$$\frac{1}{\pi\rho_1^2(1-\alpha^2)}\int_{\alpha\rho_1}^{\rho_1}\rho d\rho\, e^{i\Gamma\rho^2}\int_0^{2\pi}e^{-i(\beta/2)\cos 2\phi}\,d\phi = \frac{1}{1-\alpha^2}\int_{\alpha^2}^{1}e^{i\lambda t}\,J_0(\mu t)\,dt, \quad . \quad . \quad (2)$$

where $\lambda = \Gamma\rho_1^2$ and $4\mu d^4 = \kappa\rho_1^2\gamma_1^2\sigma_4$ as before. The intensity distribution is clearly symmetrical about the mid-point of the astigmatic separation, *i.e.*, about the point $\lambda = 0$; the intensity at this point is a maximum and is given by

$$\sqrt{I} = \frac{1}{1-\alpha^2}\int_{\alpha^2}^{1}J_0(\mu t)\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while the intensity at either principal focus is given by the squared modulus of

$$\frac{1}{1-\alpha^2}\int_{\alpha^2}^{1}e^{\pm i\mu t}J_0(\mu t)\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As a numerical example let us consider the value of I from (3) when $\alpha\sqrt{2} = 1$, $\mu = 1\cdot 2$; this value of μ was found to lead to the Rayleigh limit in the full aperture case. I find that

$$\sqrt{I} = 2\int_{\cdot 5}^{1}J_0(1\cdot 2t)\,dt = 0\cdot 8022 \text{ approximately, so that } I = 0\cdot 6435.$$

The Rayleigh limit of $0\cdot 80$ is therefore not satisfied and too large a value of μ has been taken. The annular aperture is seen to be unfavourable in the case of astigmatism—at least as far as the central intensity is considered. This result might have been anticipated from the form of (3); for as $\alpha \to 1$ the value of $\sqrt{I} \to J_0(\mu)$ and the value $\mu = 0\cdot 661$ brings us here to the Rayleigh limit. The limiting value of μ for the annular aperture lies, therefore, between $1\cdot 2$ in the full aperture case and $0\cdot 66$ in the limiting case when the aperture becomes a narrow ring. The intensity distribution away from the central line is considered in §§ 11 *et seq.* for the limiting case in which $\alpha \to 1$.

§ 5. It has been shown that the effect of 'distortion' is merely to move the diffraction pattern as a whole; the results, therefore, obtained in § 1 may be applied at once to this case.

§ 6. *The Effect of the Combined Geometrical Aberrations.*—Hitherto the diffraction effect of the separate aberrations only has been considered. In certain cases, however, the effect of the combined aberrations may be investigated simply. Let us consider that the exit pupil is an annulus of small width; then, with the usual notation, the intensity at the pole upon the receiving plane is given by the squared modulus of the expression

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\kappa(\mathrm{U}_1+r_0)}\,d\phi. \qquad (1)$$

Using the general expression, Part II, § 6, the integration terms in $\kappa(\mathrm{U}_1+r_0)$ may be written

$$\kappa(\mathrm{U}_1+r_0) = -z\cos(\phi-\phi') - \frac{\kappa \mathrm{XY}_1\rho_1}{d^2}\cos\phi + \frac{\kappa\sigma_2\rho^3\mathrm{Y}_1}{2d^4}\cos\phi + \frac{\kappa\sigma_5\rho\mathrm{Y}_1^3}{2d^4}\cos\phi - \frac{\kappa\sigma_4\rho^2\mathrm{Y}_1^2}{2d^4}\cos^2\phi$$

$$= -\zeta\cos(\phi-\psi) - 2\mu\cos^2\phi, \qquad (2)$$

where

$$\zeta\cos\psi = z\cos\phi' + \frac{\kappa\mathrm{XY}_1\rho}{d^2} - \frac{\kappa\sigma_2\rho^3\mathrm{Y}_1}{2d^4} - \frac{\kappa\sigma_5\rho\mathrm{Y}_1^3}{2d^4} = z\cos\phi' - \mathrm{A}\ (\text{say}),$$

and

$$\zeta\sin\psi = z\sin\phi'$$

so that

$$\tan\psi = \frac{z\sin\phi'}{z\cos\phi'-\mathrm{A}}, \quad \zeta^2 = z^2 - 2\mathrm{A}z\cos\phi' + \mathrm{A}^2.$$

As usual $4\mu d^2 = \kappa\sigma_4\rho^2\mathrm{Y}_1^2$. The quantity ζ denotes distance measured from the point (A, 0). The general expression (1) becomes now

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\zeta\cos(\phi-\psi)-2i\mu\cos^2\phi}\,d\phi, \qquad (3)$$

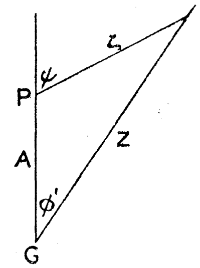
and the squared modulus of this expression will give the light intensity in the presence of all the first-order geometrical aberrations and indeed of spherical aberration of any order.

Assume now that $\mu = 0$, *i.e.*, that the system is free from astigmatism; then (3) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\zeta\cos(\phi-\psi)}\,d\phi = \mathrm{J}_0(\zeta)$$

so that $\sqrt{\mathrm{I}} = \mathrm{J}_0(\zeta)$. The diffraction pattern is, therefore, a series of concentric circles with the point (A, 0) as common centre, and the radius of the first dark ring is 2·405 and that of the first bright ring is 3·832. The effect of the first-order geometrical aberrations (apart from astigmatism) is thus to move the diffraction pattern bodily a distance A from its position for an aberrationless system, and the distance A depends upon the geometrical aberrations present. It is seen that a decided advantage is conferred for the resolution of double stars, for the radius of the first dark ring in the AIRY disc is 3·832 as compared with 2·405 with the narrow annular aperture.

§ 7. The geometrical meaning of ζ is made clear by the diagram. G is the Gaussian image point or its projection upon the receiving plane, z, ϕ' are the usual polar co-ordinates, so that ζ, ψ are polar co-ordinates with P as new origin and the same initial line.



The general value of (3) § 6 may be evaluated as in the Appendix, but it is sufficient to consider the two cases in which $\psi = 0, \frac{\pi}{2}$ respectively. Putting $\psi = \frac{\pi}{2}$ in (3) we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\zeta \sin\psi - 2i\mu\cos^2\phi}\,d\phi = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-2i\mu)^n}{\underline{|n}}\int_0^{2\pi} e^{-i\zeta\sin\phi}\cos^{2n}\phi\,d\phi$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}(-2i\mu)^n\,\frac{\underline{|2n}}{2^n\,(\underline{|n})^2}\,\frac{J_n(\zeta)}{\zeta^n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from the formulæ

$$\int_0^{\pi} e^{\pm iz\cos\phi}\sin^{2n}\phi\,d\phi = \int_0^{\pi} e^{\pm iz\sin\phi}\cos^{2n}\phi\,d\phi = \frac{\pi\,\underline{|2n}}{2^n\,\underline{|n}}\,\frac{J_n(z)}{z^n}.$$

If $\psi = 0$ expression (3) leads to a consideration of

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\zeta\cos\phi + 2i\mu\sin^2\phi}\,d\phi = \sum_{n=0}^{\infty}(2i\mu)^n\,\frac{\underline{|2n}}{2^n\,(\underline{|n})^2}\,\frac{J_n(\zeta)}{\zeta^n}\ . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

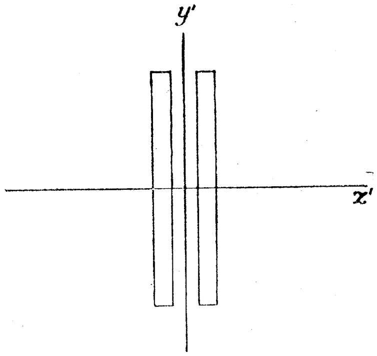
Expressions (1), (2) are seen to lead to the same result since the light intensity is given, in each case, by the squared modulus of the expression; either may be written

$$\sum_{n=0}^{\infty}\ {}_{2n}C_n\,\frac{J_n(\zeta)}{\zeta^n}\,(i\mu)^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Upon the central line of the contours we have from the general integral, by putting $\zeta = 0$, the expression

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-2i\mu\cos^2\phi}\,d\phi = J_0(\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $\mu = 0·661$ the central intensity is 0·80 times its value for an aberrationless system. Substituting this value of μ in (3) above I find that the radius of the first dark ring is about 2·6, in the usual system of co-ordinates, as compared with the value 2·4 if $\mu = 0$, *i.e.*, if astigmatism be absent.



§ 8. *The Rectangular Aperture.*—Let the exit pupil be two parallel rectangular apertures symmetrically placed with respect to the axis of y' and extending from $y' = +a$ to $y' = -a$. Let the breadth be given by $z' = b'$ to $z' = b$ and $z' = -b'$ to $z' = -b$. Here S, the area of the exit pupil, is given by $S = 4a\,(b - b')$, so that neglecting aberrations and taking an object point upon the axis of the instrument, as we may do without loss of generality, we have the expression

$$\frac{1}{4a\,(b-b')}\iint e^{-i\kappa(\rho\rho'/d)\cos(\phi-\phi')}\,dy'dz' = \frac{1}{4a\,(b-b')}\iint e^{-i(\kappa/d)(y'h+z'k)}\,dy'dz' \quad . \quad . \quad (1)$$

if h, k be the co-ordinates of the pole at which the light intensity is required. Expression (1) takes the form

$$\frac{\sin\left(\frac{\kappa ah}{d}\right)}{\left(\frac{\kappa ah}{d}\right)}\left\{\frac{\sin\left(\frac{\kappa bk}{d}\right)}{\frac{\kappa k}{d}} - \frac{\sin\left(\frac{\kappa b'k}{d}\right)}{\frac{\kappa k}{d}}\right\}\frac{1}{b-b'} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, therefore, we consider only the effect upon a line in the receiving plane parallel to the y-axis above—and this is the general case of practical importance—or if we assume that b, b' are small compared with a, then the light intensity I is given by

$$\sqrt{\mathrm{I}} = \frac{\sin\left(\frac{\kappa ah}{d}\right)}{\left(\frac{\kappa ah}{d}\right)} = \frac{\sin z}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where z is the usual co-ordinate. An expression of precisely the same form is obtained if an aperture of only one symmetrical slit be considered.

From (3) it is evident that the minima occur where $z = n\pi$ and n is any positive or negative integer, and that maxima occur at the points given by the roots of the equation

$$z - \tan z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Moreover, at the minima the illumination is zero. Now with the usual circular aperture the minima and maxima are given respectively by the roots of the equations

$$\mathrm{J}_1(z) = 0, \qquad \mathrm{J}_2(z) = 0.$$

Comparing the minima we have for the rectangular aperture

3·142, 6·283, 9·425,

while for the circular aperture the corresponding values are

3·832, 7·016, 10·173.

For the maxima they are

4·494, 7·725, 10·943

and

5·135, 8·417, 11·620 respectively.

The rectangular aperture, therefore, confers a very distinct advantage in the resolution of close double stars, provided always that the loss of light due to the smaller aperture be a not insuperable difficulty.

§ 9. Let us consider now the effect of simple out-of-focus conditions in the absence of other aberrations. Here

$$U_1 + r_0 = \frac{X\rho^2}{2d^2} - \frac{\rho\rho'}{d}\cos(\phi - \phi'),$$

and the general integral becomes

$$\frac{1}{2a}\int e^{\frac{i\kappa X\rho^2}{2d^2} - \frac{i\kappa\rho\rho'}{d}\cos(\phi - \phi')}\,dy,$$

so that the light intensity I is given by the squared modulus of the expression

$$\int_0^1 e^{i(\mu v^2 - zv)}\,dv, \qquad (1)$$

where z is the usual co-ordinate and μ denotes the displacement of the receiving plane. It has been assumed, as usual in connection with the slit aperture, that $\phi' = 0$, so that the intensity is considered only along a line parallel to the slit. Putting $z = 0$, (1) becomes

$$\int_0^1 e^{i\mu v^2}\,dv,$$

so that if I be the intensity at a point upon the axis distant $\mu = 2\pi n$ from the paraxial image plane,

$$\sqrt{\mathrm{I}} = \left|\frac{1}{2\sqrt{n}}\int_0^{2\sqrt{n}} e^{\frac{i\pi v^2}{2}}\,dv\right|, \qquad (2)$$

the value of which may be read off at once from tables of FRESNEL'S integrals. For points away from the axis of the system (1) becomes

$$\sqrt{\mathrm{I}} = \left|\frac{1}{2\sqrt{n}}\int_{\frac{N}{\sqrt{n}}}^{2\sqrt{n}+\frac{N}{\sqrt{n}}} e^{\frac{i\pi t^2}{2}}\,dt\right|, \qquad (3)$$

by means of the substitution $v\sqrt{\mu} - \frac{z}{2\sqrt{\mu}} = t\sqrt{\frac{\pi}{2}}$ and we have written $z = 2\pi N$.

This is identical in form with Part II, § 10 (2). It also may be evaluated numerically from tables of FRESNEL's integrals.

The presence of spherical aberration of order s is denoted by the addition of a term depending upon ρ^{2s+2} to the function $U_1 + r_0$. Thus, the intensity, at distance z from the axis and in the presence of aberrations of all orders, is given by the squared modulus of

$$\int_0^1 e^{i(-zv+\mu v^2+\nu v^4+\xi v^6+\ldots)}\, dv. \quad \ldots \quad (4)$$

Let first-order aberration only be present, *i.e.*, let $\xi = \ldots = 0$; then upon the axis we have to consider

$$\int_0^1 e^{i(\mu v^2+\nu v^4)}\, dv = \int_0^1 \sum_{n=0}^{\infty} \frac{y_n}{\underline{|n}} v^{2n} dv = \sum_0^{\infty} \frac{y_n}{\underline{|n}} \frac{1}{2n+1}, \quad \ldots \quad (5)$$

where

$$y_n = (i\mu)^n + \frac{n^2}{2}(i\mu)^{n-2}\frac{i\nu}{2} + \ldots + \frac{n^{2\nu}}{2\,.\,4\ldots r}(i\mu)^{n-2\nu}\left(\frac{i\nu}{2}\right)^r + \ldots.$$

§ 10. *Numerical Results.*—In simple out-of-focus conditions we have the integral (2): from tables of FRESNEL's integrals it is found that if $n = 0{\cdot}25$, then

$$I = \left| \int_0^1 e^{i\pi v^2/2}\, dv \right|^2 = 0{\cdot}8319,$$

so that $n = \frac{1}{4}$ brings us practically to the RAYLEIGH limit; this, then, is the maximum shift of focus plane permissible. Again, for a point $z\ (= 2\pi N)$ upon the plane, from (2) we have

$$\sqrt{I} = \left| \int_{2N}^{1+2N} e^{i\pi v^2/2}\, dt \right|.$$

From the FRESNEL tables I find the following values for the intensity :—

N =	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8.
I =	0·8319	0·6486	0·4837	0·3251	0·1902	0·0883	0·0344	0·0089	0·0207.

The first maximum is given, therefore, by $N = 0{\cdot}73$, approximately, and its radius is $2\pi \times 0{\cdot}73$, *i.e.*, $4{\cdot}6$.

From § 9 (5) we have that the axial intensity in the presence of first-order aberration given by $\nu = 2\pi n_1$, together with out-of-focus conditions given by $\mu = 2\pi n$, is given by the squared modulus of

$$\int_0^1 e^{2\pi i(nv^2+n_1v^4)}\, dv$$

I find that $n_1 = \frac{7}{4}$, $n = -\frac{7}{4}$ gives approximately the RAYLEIGH limit, so that if change of focus be admitted $1\frac{3}{4}$ wave-lengths of first-order spherical aberration are permissible. It will be remembered that in the case of the full circular aperture one wave-length only of first-order aberration was permissible in order to conform to the RAYLEIGH limit.

§ 11. If coma be present alone, then the terms in $U_1 + r_0$ are

$$U_1 + r_0 = \frac{\sigma_2}{2}\frac{Y_1\rho^3}{d^4}\cos\phi - \frac{\rho\rho'}{d}\cos\phi, \qquad \text{where we have put } \phi' = 0,$$

$$= \frac{\sigma_2}{2}\frac{Y_1}{d^4}y'(y'^2 + z'^2) - \frac{\rho' y'}{d},$$

in the notation already used. For a narrow slit we may neglect the z' dimension, and we are led to an expression of the form

$$\int_{-a}^{+a} e^{iAy^3 - iBy}\,dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A, B are constants; since the function in the index of the exponential is odd, this becomes

$$\int_0^a \cos(Ay^3 - By)\,dy,$$

so that in the usual notation we have

$$\sqrt{I} = \int_0^1 \cos(\beta u^3 - zu)\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This expression gives the light intensity at any point z upon a line passing through the Gaussian image point and parallel to the aperture, in the presence of coma of amount β. The dark points upon the line are given by the roots of the equation in z

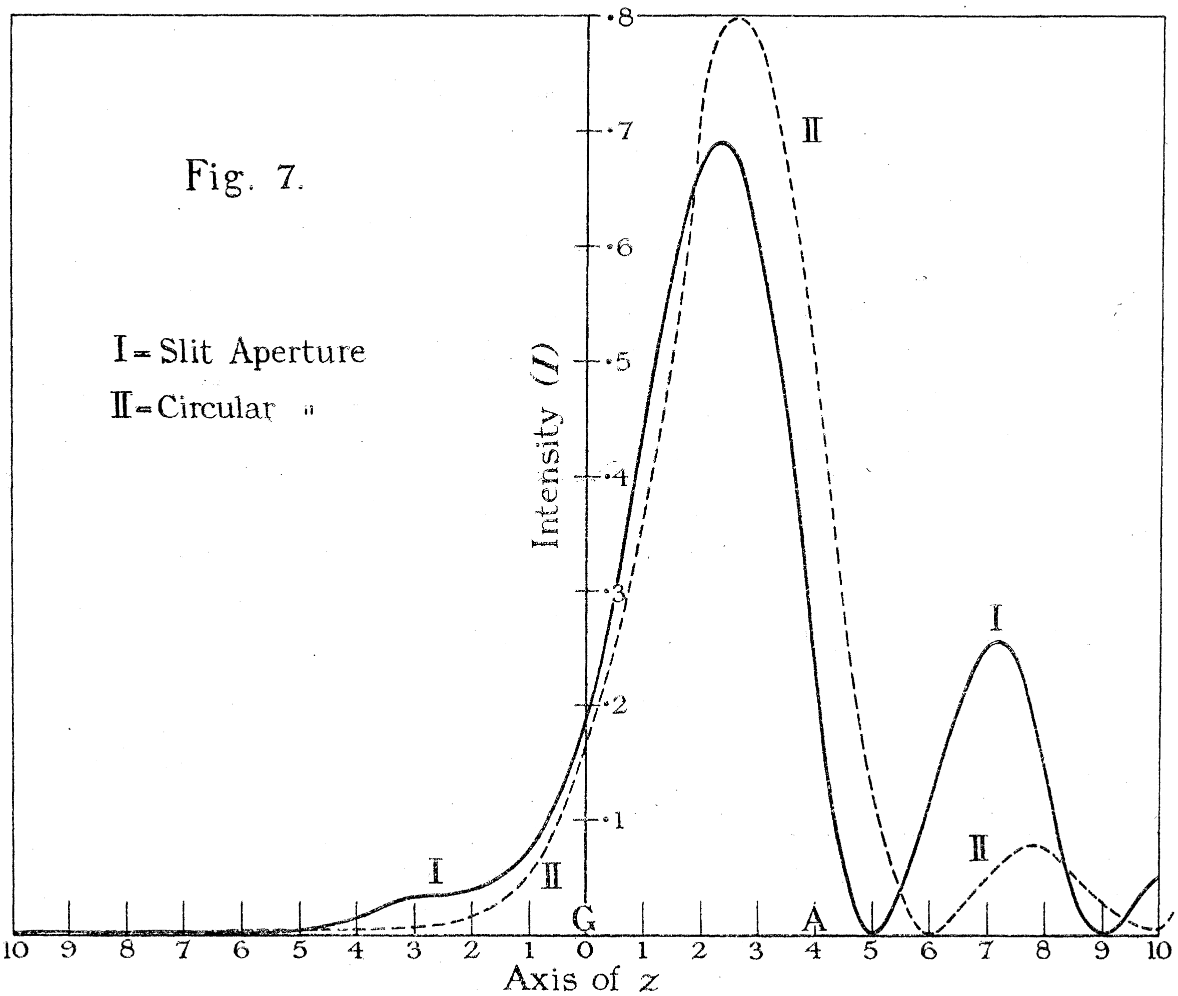
$$\int_0^1 \cos(\beta u^3 - zu)\,du = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while the positions of the turning values of I, and these in general will be maxima, are given by the roots of the equation

$$\int_0^1 \sin(\beta\sqrt{u^3} - z\sqrt{u})\,du = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Numerical Results.—By mechanical quadratures of expression (2), I find that, in the case $\beta = 3\cdot25$, the maximum value of the intensity is obtained when $z = 1\cdot75$, approximately, and that it is then $0\cdot7808$ times the intensity at the Gaussian image in the absence of coma; this amount of coma, therefore, brings us to the RAYLEIGH limit for this aperture. If $\beta = 4$—and this brings us to the limit for the full circular aperture—then the maximum intensity sinks to $0\cdot6894$, at the point $z = 2\cdot35$. It is seen that the maximum intensity occurs at the same point, approximately, relative to the geometrical coma figure as in the circular aperture case.

The diagram given (fig. 7) shows the intensity distribution along the line in the image plane parallel to the aperture, in the case $\beta = 4$; and the dotted curve shows the



Light Intensity along Axis of Coma Figure in presence of Coma of amount $\beta = 4$.

I. Slit Aperture: $\sqrt{\mathrm{I}} = \int_0^1 \cos(4u^3 - zu)\, du.$

II. Full Circular Aperture (dotted curve): $\sqrt{\mathrm{I}} = \int_0^1 J_0(4\sqrt{u^3} - z\sqrt{u})\, du.$

The numerical values of the intensities which are shown graphically in the diagram (Curve I) are are follows:—

z	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
Intensity	0·0021	0·0006	0·0003	0·0012	0·0000	0·0040	0·0163	0·0336	0·0380	0·0709

z	0	1	2	3	4	5	6	7	8	9	10.
Intensity	0·1859	0·4374	0·6708	0·5955	0·2258	0·0009	0·1180	0·2543	0·1317	0·0023	0·0510.

The maximum value of the intensity is 0·69 (approximately) and occurs at the point $z = 2 \cdot 35$.

distribution along the same line when the aperture is a circle and the same amount of coma is present. It is seen that for the resolution of double stars a distinct advantage is conferred by the slit aperture; for the radius of the dark 'ring' in the direction away from the head of the geometrical coma figure is 2·75 in this case, as compared with 3·35 for the full circular aperture. It is interesting to notice that in the AIRY disc this same radius becomes 3·83. For the first light 'ring' the figures are:—slit aperture, 4·75; full aperture, 5·35; AIRY disc, 5·14.

§ 12. If curvature and astigmatism be present, then

$$U_1 + r_0 = \frac{\rho^2}{2d^2}\left\{X - \frac{Y_1^2}{2d^2}\overline{\sigma_3 + 2\sigma_4}\right\} - \frac{\rho}{d}\left\{\rho' + X\frac{Y_1}{d}\right\},$$

where we have put $\phi' = 0$, $\phi = 0$; so that on the central line, given by $\rho' d + XY_1 = 0$, we have an integral of the form

$$\int_{-a}^{a} e^{i\Lambda\rho^2}\, d\rho,$$

and this reduces in the usual way to

$$\int_0^1 e^{i\lambda u^2}\, du, \quad \ldots \ldots \ldots \quad (1)$$

where λ, as usual, measures the distance from the primary focus.

If the slit aperture be now rotated through a right-angle, *i.e.*, if we put $\phi = \pi/2$, then the expression obtained is

$$\int_0^1 e^{i\lambda' u^2}\, du, \quad \ldots \ldots \ldots \quad (2)$$

where λ' measures the distance from the secondary focus. Away from the central line along a line parallel to the slit we have the expression

$$\int_0^1 e^{i\lambda u^2 - izu}\, du. \quad \ldots \ldots \ldots \quad (3)$$

This is of the same form as § 9 (1), and therefore the discussion given will apply here also. Indeed, it is evident that the function of the slit aperture is merely to eliminate rays travelling to the focal line parallel to it; simple out-of-focus conditions are, therefore, reproduced.

§ 13. The effect of distortion is merely to move the diffraction pattern as a whole, and thus the results of § 8 may be quoted, together with those of § 9, for out-of-focus effects.

§ 14. *The Semi-Circular Aperture.*—Let the exit pupil be semi-circular in form; then, for an aberrationless system we have, as usual, $\kappa(U_1 + r_0) = -\kappa(\rho\rho'/d)\cos(\phi - \phi')$; if, therefore, we write $td = \kappa\rho\rho'$ we have to consider the expression

$$\int_0^{\rho_1} \rho\, d\rho \int_0^{\pi} e^{-it\cos(\phi-\phi')}\, d\phi, \quad \ldots \ldots \ldots \quad (1)$$

where ρ_1 is the radius of the exit pupil. Here it is assumed that the Gaussian image point lies upon the projection in the receiving plane of the diameter of the exit pupil, *i.e.*, that $0 \leq \phi \leq \pi$.

Now

$$\sin(x \sin\theta) = 2 \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1)\theta,$$

$$\cos(x \sin\theta) = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta,$$

so that

$$\int_0^{\pi} e^{-it\cos(\phi-\phi')}\, d\phi = \int_0^{\pi} \cos(t \cos\overline{\phi-\phi'})\, d\phi - i \int_0^{\pi} \sin(t\cos\overline{\phi-\phi'})\, d\phi$$
$$= \pi J_0(t) - 4i\eta(t), \quad \ldots \quad (2)$$

where

$$\eta(t) = J_1(t)\sin\phi' - \frac{J_3(t)\sin 3\phi'}{3} + \frac{J_5(t)\sin 5\phi'}{5} - \ldots \quad \ldots \quad (3)$$

Write, as usual, $zd = \kappa\rho_1\rho'$; then the expression whose squared modulus gives the intensity is

$$\frac{2}{z^2}\int_0^z \{\pi J_0(t) - 4i\eta(t)\}\, t\, dt. \quad \ldots \quad (4)$$

It is seen that this reduces to unity at the Gaussian image point, where $z = 0$; we have, therefore,

$$\frac{2J_1(z)}{z} - \frac{8i}{\pi z^2}\int_0^z \eta(t)\, t\, dt,$$

i.e.,

$$\frac{2J_1(z)}{z} - \frac{8i}{\pi z^2}\zeta(z), \quad \ldots \quad (5)$$

if $\zeta(z) = \int_0^z \eta(t)\, t\, dt$. Thus

$$I^2 = \left\{\frac{2J_1(z)}{z}\right\}^2 + \frac{64}{\pi^2 z^4}\zeta^2(z). \quad \ldots \quad (6)$$

§ 15. If $\phi' = 0$, *i.e.*, if the pole lie upon the edge of the projection of the aperture upon the receiving plane, then $\eta(t) = 0$; so that (6) above reduces to

$$I^2 = \left\{\frac{2J_1(z)}{z}\right\}^2,$$

and this is the AIRY disc. If $\phi' = \pi/2$, then

$$\eta(t) = J_1(t) + \frac{J_3(t)}{3} + \frac{J_5(t)}{5} + \ldots$$

so that

$$\zeta(z) = \int_0^z \eta(t)\, t\, dt = z\sum_{n=1}^{\infty}\frac{J_{2n}(z)}{2n-1} + \int_0^z\left(\frac{1-J_0(t)}{t}\right)dt, \quad \ldots \quad (1)$$

since

$$\int_0^z \frac{J_{2n-1}(t)}{2n-1}\, t\, dt = \frac{zJ_{2n}(z)}{2n-1} - \int_0^z J_{2n}(t)\, dt, \quad \text{and} \quad 1 - J_0(z) = 2\sum_{n=1}^{\infty} J_{2n}(z).$$

Thus (1) combined with (6) of the preceding paragraph gives the light intensity at distance z from the Gaussian image point upon the line $\phi' = 0$. Formulæ equivalent to these have been given by BRUNS, 'Astron. Nachr.,' vol. civ, p. 1 (1883).

It is worthy of notice that the maxima and minima of the intensity are at the points given by the roots of the equation

$$\frac{\pi^2}{16}\frac{J_1(z)\,J_2(z)}{\zeta(z)} = \frac{d}{dz}\left\{\frac{\zeta(z)}{z^2}\right\}. \quad \ldots \ldots \quad (2)$$

From §§ 14, 15 it is evident that the effect of the aperture is to scatter light into the outer parts of the field. There will be no dark points now upon the line $\phi = \pi/2$, nor, indeed, anywhere except upon the line $\phi' = 0$ (or π).

§ 16. If spherical aberration or out-of-focus conditions be present, then the above investigation is modified by the introduction of a factor of the form Exp. $(i\mu t^2 + it\nu^9 + \ldots)$; so that we have to consider the expression

$$\frac{2}{\pi z^2}\int_0^z [\pi J_0(t) - 4i\eta\,(t)]\, e^{i\mu t^2 + i\nu t^4 + \ldots}\, t\, dt. \quad \ldots \ldots \quad (1)$$

§ 17. If coma be present, then upon the Gaussian image plane,

$$U_1 + r_0 = -\frac{C\rho}{d}\cos(\phi - \psi) \ldots \text{App. § 6.}$$

We have to consider, therefore, the expression

$$\int_0^{\rho_1} \rho d\rho \int_0^{\pi} e^{-\frac{i\kappa C\rho}{d}\cos(\phi-\psi)}\, d\phi,$$

i.e.,

$$\frac{2}{\pi z^2}\int_0^{\rho_1} \rho d\rho \left[\pi J_0\left(\frac{\kappa C\rho}{d}\right) - 4i\eta\left(\frac{\kappa C\rho}{d}\right)\right],$$

and by means of the substitution $\kappa\rho\rho' = zv^{\frac{1}{2}}d$ we have

$$\int_0^1 J_0(u)\, dv + \frac{4i}{\pi}\int_0^1 \eta\,(u)\, dv, \quad \ldots \ldots \quad (1)$$

where $u^2 = z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3$ and β, as usual, is the measure of the coma present. The light intensity is given by the squared modulus of (1). Upon the axis of the coma-figure $\phi' = 0$, and then $\eta\,(u) = 0$, so that

$$\sqrt{I} = \int_0^1 J_0\,(z\sqrt{v} - \beta\sqrt{v^3})\, dv, \quad \ldots \ldots \quad (2)$$

and this is the same formula as with the full circular aperture. At right angles to the axis of the coma-figure $\phi' = \pi/2$, and then (1) becomes

$$\int_0^1 J_0\,(\sqrt{z^2 v + \beta^2 v^3}) + \frac{4i}{\pi}\int_0^1 \eta\,(\sqrt{z^2 v + \beta^2 v^3})\, dv. \quad \ldots \ldots \quad (3)$$

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Here the effect of the closing down of the aperture is seen in the presence of the second integral.

§ 18. In the above it has been assumed that the Gaussian image point lies upon the projection upon the receiving plane of the diameter of the exit pupil, *i.e.*, that $0 \leq \phi \leq \pi$. Assume now that the image point lies upon a line perpendicular to this, *i.e.*, that $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. Then we are led to the formula

$$\int_0^{\rho_1} \rho d\rho \int_{-\pi/2}^{\pi/2} e^{-\frac{i\kappa c\rho}{d}\cos(\phi-\psi)} d\phi. \quad \ldots \ldots \quad (1)$$

Now

$$\int_{-\pi/2}^{\pi/2} e^{it\cos(\phi-\phi')} d\phi = \pi J_0(t) + 4i\left[\frac{J_1(t)}{1}\cos\phi' + \frac{J_3(t)}{3}\cos 3\phi' + \frac{J_5(t)}{5}\cos 5\phi' + \ldots\right]$$
$$= \pi J_0(t) + 4i\eta_1(t) \text{ (say).}$$

Then the expression whose squared modulus gives the light intensity in this case is

$$\int_0^1 J_0(u)\, du - \frac{4i}{\pi}\int_0^1 \eta_1(u)\, du,$$

where $u^2 = z^2 v - 2z\beta v^2 \cos\phi' + \beta^2 v^3$. If $\phi' = 0$ then

$$\eta_1(t) = \frac{J_1(t)}{1} + \frac{J_3(t)}{3} + \frac{J_5(t)}{5} + \ldots,$$

and we have

$$\int_0^1 J_0(z\sqrt{v} - \beta\sqrt{v^3})\, dv - \frac{4i}{\pi}\int_0^1 \eta_1(z\sqrt{v} - \beta\sqrt{v^3})\, dv, \quad \ldots \quad (2)$$

while if $\phi' = \pi/2$ then $\eta_1(t) = 0$, and we have

$$\int_0^1 J_0(\sqrt{z^2 v + \beta^2 v^3})\, dv. \quad \ldots \ldots \quad (3)$$

§ 19. *Effect of Curvature and Astigmatism.*—Curvature and astigmatism are governed by the coefficients σ_3, σ_4. If curvature alone be present the effects are simply out-of-focus effects, and the results of § 8 may be quoted. With the introduction of astigmatism, however, a new integral presents itself. From Appendix § 9 we have the expression

$$\int_0^{\rho_1} \rho d\rho\, e^{i\gamma\rho^2} \int_{-\pi/2}^{\pi/2} e^{-i\zeta\cos(\phi-\psi) + i\beta\sin^2\phi}\, d\phi$$

in the notation employed there. The limits of integration are now $-\pi/2$, $\pi/2$ instead of 0, 2π. By following the method of that paragraph we are led to a consideration of the integral

$$\int_{-\pi/2}^{\pi/2} e^{-i\zeta\cos\phi\cos\psi} \sin^{2n}\phi\, d\phi = 2\int_0^{\pi/2} e^{-i\zeta\cos\phi\cos\psi} \sin^{2n}\phi\, d\phi$$
$$= \frac{2^n \Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{(\zeta\cos\psi)^n}\{J_n(\zeta\cos\psi) - iH_n(\zeta\cos\psi)\},$$

where $H_n(x)$ is STRUVE's function of order n. As the case of most importance we may put $\psi = 0$ and consider the light distribution upon the ' primary ' plane, *i.e.*, put $\nu = 0$, in the notation of Appendix § 9. Then the expression whose squared modulus will give the light intensity is

$$\frac{2}{z^2}\sum_{n=0}^{\infty}\frac{1}{2^n \lfloor n}\, Y'_{2n}\,(U_{n+1} - iV_{n+1}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$U_{n+1} = \int_0^{z_1} z^{n+1} J_n(z)\,dz = J_{n+1}(z_1)$$

and

$$V_{n+1} = \int_0^{z_1} t^{n+1} H_n(t)\,dt = H_{n+1}(z_1).$$

Here the effect of closing down the aperture is seen in the presence of the function $H_n(z)$.

§20. As shown in Part II the effect of distortion is merely to move bodily the whole diffraction pattern produced by an aberrationless system. No new investigation is needed, therefore, beyond that given in § 14.

APPENDIX.

§ 1. The light intensity at any point is proportional to the squared modulus of $\int e^{i\kappa(U_1 + r_0)}\,ds$, where the integration is taken over the exit pupil, and, from § 6 (1),

$$U_1 + r_0 = -\frac{Y_1^2}{2m} + \frac{Y_1^4}{8m}\frac{(s^3 - m)}{d^3} - \Phi_4 + \frac{Y_1\rho'}{d}\cos\phi'$$

$$-\frac{\rho\rho'}{d}\cos(\phi - \phi') + \frac{X}{2d^2}(Y_1^2 + \rho^2 - 2Y_1\rho\cos\phi), \quad . \quad . \quad (1)$$

and

$$\Phi_4 = \frac{1}{8d^4}[\sigma_1\rho^4 - 4\sigma_2\rho^3 Y_1\cos\phi + 2\sigma_3\rho^2 Y_1^2 + 4\sigma_4\rho^2 Y_1^2\cos^2\phi - 4\sigma_5\rho Y_1^3 + \sigma_6 Y_1^4]. \quad . \quad (2)$$

The variables of integration are (ρ, ϕ), and if the light intensity at the Gaussian image point be taken to be unity the relative intensity I at any other point is given by

$$\sqrt{I} = \left|\frac{1}{S}\int e^{i\kappa(U_1 + r_0)}\,dS\right|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where S is the area of the exit pupil. This is the general expression to be considered and various specialisations may be made ; for example, in the first place, the exit pupil will be taken to be circular and the effect of the various aberrations will be considered separately. Afterwards other types of exit pupil may be considered.

§2. Let all the aberration coefficients vanish except σ_1; then the integration terms in (1) are given by

$$\mathrm{U}_1 + r_0 = -\frac{\sigma_1}{8}\left(\frac{\rho}{d}\right)^4 + \frac{\mathrm{X}}{2}\left(\frac{\rho}{d}\right)^2 - \mathrm{X}\frac{\rho \mathrm{Y}_1}{d^2}\cos\phi - \frac{\rho\rho'}{d}\cos(\phi - \phi')$$

$$= -\frac{\sigma_1}{8}\left(\frac{\rho}{d}\right)^4 + \frac{\mathrm{X}}{2}\left(\frac{\rho}{d}\right)^2 - \frac{\rho \mathrm{C}}{d}\cos(\phi - \psi), \quad \ldots \ldots \quad (1)$$

where

$$\mathrm{C}^2 = \left(\mathrm{X}\frac{\mathrm{Y}_1}{d} + \rho'\cos\phi'\right)^2 + (\rho'\sin\phi')^2, \quad \tan\psi = \frac{\rho' d\sin\phi'}{\mathrm{XY}_1 + \rho' d\cos\phi'}.$$

The terms omitted give modulus unity in §1 (3) and therefore add nothing to the result. Without loss of generality in the form of the result we may write $\mathrm{Y}_1 = 0$.

Write now

$$\frac{\kappa\rho\rho'}{d} = t, \quad \frac{\kappa \mathrm{X}}{2}\left(\frac{\rho}{d}\right)^2 = \mu t^2, \quad \frac{\kappa\sigma_1}{8}\left(\frac{\rho}{d}\right)^4 = -\nu t^4, \quad \text{and let } \frac{\kappa\rho_1\rho'}{d} = z$$

where ρ_1 is the radius of the exit pupil. Then

$$\kappa(\mathrm{U}_1 + r_0) = \mu t^2 + \nu t^4 - t\cos(\phi - \phi'),$$

so that

$$\sqrt{\mathrm{I}} = \left|\frac{1}{\pi z^2}\int_0^z t\,dt\int_0^{2\pi} e^{i\mu t^2 + i\nu t^4 - it\cos\overline{\phi - \phi'}}\,d\phi\right|$$

$$= \left|\frac{2}{z^2}\int_0^z e^{i\mu t^2 + i\nu t^4}\,t\mathrm{J}_0(t)\,dt\right|, \quad \ldots \ldots \quad (2)$$

$$\textit{i.e.,} \quad \sqrt{\mathrm{I}} = \left|\int_0^1 e^{i\mu t + i\nu t^2}\,\mathrm{J}_0(z\sqrt{t})\,dt\right|. \quad \ldots \ldots \quad (3)$$

§3. The integral §2 (2) is of the form

$$\int_0^z f(t^2)\,t\mathrm{J}_0(t)\,dt = \int_0^z f(t^2)\frac{d}{dt}\{t\mathrm{J}_1(t)\}\,dt$$

$$= f(z^2)\,z\mathrm{J}_1(z) - \int_0^z 2f_1(t^2)\frac{d}{dt}\{t^2\mathrm{J}_2(t)\}\,dt$$

$$= \ldots \ldots \ldots$$

$$= f(z^2)\,z\mathrm{J}_1(z) - 2f_1(z^2)\,z^2\mathrm{J}_2(z) + 2^2 f_2(z^2)\,z^3\mathrm{J}_3(z) - \ldots$$

$$+ (-1)^n\,2^n f_n(z^2)\,z^{n+1}\mathrm{J}_{n+1}(z) + \ldots \quad \ldots \quad (1)$$

where

$$f_{n+1}(t^2) = \frac{d}{dt^2}f_n(t^2), \quad \textit{i.e.,} \quad f_{n+1}(x) = \frac{d}{dx}f_n(x), \quad \text{if} \quad x = t^2.$$

Let* $\quad f(x) = e^{i\mu x + i\nu x^2}$ so that $f_1(x) = i(\mu + 2\nu x)f(x)$.

* The method is applicable to the case including spherical aberration of any order.

Differentiating r times with respect to x,

$$f_{r+1}(x) = i(\mu + 2\nu x) f_r(x) + 2i\nu r f_{r-1}(x);$$

assume now $f_r(x) = \dfrac{u_r}{x^r}$, so that

$$u_{r+1} = i(\mu x + 2\nu x^2) u_r + 2i\nu x^2 u_{r-1}.$$

Now in (1) t will assume its limiting value z, and therefore we may write

$$\mu x = 2\pi n, \quad 2\nu x^2 = 4\pi n,$$

and then u_{r+1} is given by the difference equation

$$u_{r+1} + \alpha u_r + \beta r u_{r-1} = 0, \qquad \ldots \qquad (2)$$

where $\alpha = -2\pi(n + 2n_1)$, $\beta = -4\pi n_1$. Thus u_r is independent of (ρ', ϕ').

Substituting in § 2 (2),

$$\sqrt{\mathrm{I}} = 2\frac{J_1(z)}{z} u_0 - 2^2 \frac{J_2(z)}{z^2} u_1 + \ldots + (-1)^n 2^{n+1} \frac{J_{n+1}(z)}{z^{n+1}} u_n + \ldots \qquad (3)$$

and by the usual methods the solution of (2) is

$$u_r = u_0 \left[\mu_1{}^r - \mu_1{}^{r-2} \beta \frac{r^{(2)}}{2} + \mu_1{}^{r-4} \beta^2 \frac{r^{(4)}}{2 \,.\, 4} - \frac{\mu_1{}^{r-6} \beta^3 r^{(6)}}{2 \,.\, 4 \,.\, 6} + \ldots \right],$$

where $\mu_1 = -\alpha$ and $r^{(m)} = r \,.\, r-1 \ldots r-m+1$.

Let $u_r = (c_r + id_r) u_0 = (c_r \cos\chi - d_r \sin\chi) + i(d_r \cos\chi + c_r \sin\chi)$, since $u_0 = e^{i\chi}$, if $\chi = 2\pi n + 2\pi n_1$. Thus

$$\sqrt{\mathrm{I}} = |p + iq|,$$

where

$$p = \mathrm{U}\cos\chi - \mathrm{V}\sin\chi$$

$$q = \mathrm{V}\cos\chi + \mathrm{U}\sin\chi,$$

and

$$\mathrm{U} = \sum_{n=0}^{\infty} (-1)^n c_n 2^{n+1} \frac{J_{n+1}(z)}{z^{n+1}}, \qquad \mathrm{V} = \sum_{0}^{\infty} (-1)^n d_n 2^{n+1} \frac{J_{n+1}(z)}{z^{n+1}}, \qquad (4)$$

so that

$$\mathrm{I} = \mathrm{U}^2 + \mathrm{V}^2. \qquad \ldots \qquad (5)$$

If z be not too small the first few terms of (4) will give the value of I to a high order of accuracy: the quantities c_n, d_n may be written down at once from the solution of the difference equation, or the following may be used.

If

$$\mu_2 = 2\pi(n+2n_1), \quad \beta_2 = 4\pi n_1,$$

$$c_{2r} = (-1)^r\left[\mu_2^{2r} - \frac{(2r)^{(4)}}{2.4}\mu_2^{2r-4}\beta_2^2 + \frac{(2r)^{(8)}}{2.4.6.8}\mu_2^{2r-8}\beta_2^4 - \dots\right],$$

$$d_{2r} = (-1)^{r-1}\left[\frac{(2r)^{(2)}}{2}\mu_2^{2r-2}\beta_2 - \frac{(2r)^{(6)}}{2.4.6}\mu_2^{2r-6}\beta_2^3 + \dots\right],$$

$$c_{2r+1} = (-1)^r\left[\frac{(2r+1)^{(2)}}{2}\mu_2^{2r-1}\beta_2 - \frac{(2r+1)^{(6)}}{2.4.6}\mu_2^{2r-5}\beta_2^3 + \dots\right],$$

$$d_{2r+1} = (-1)^r\left[\mu_2^{2r+1} - \frac{(2r+1)^{(4)}}{2.4}\mu_2^{2r-3}\beta_2^2 + \dots\right]. \quad (6)$$

§ 4. The above series are useful for large values of z; for small values the series below may be used:—

$$\int_0^z f(t^2)\, t\mathrm{J}_0(t)\, dt = f_1(z^2)\,\mathrm{J}_0(z) + f_2(z^2)\frac{\mathrm{J}_1(z)}{z} + f_3(z^2)\frac{\mathrm{J}_2(z)}{z^2} + \dots, \quad (1)$$

where $2df_{n+1}/dx = f_n$ and $x = t^2$; this assumes that the constants in the successive integrations are chosen so that $f_n(0) = 0$. Assume now that $f = e^{i\mu x + i\nu x^2}$, so that if

$$f = 1 + \frac{b_1 x}{\lfloor 1} + \frac{b_2 x^2}{\lfloor 2} + \dots + \frac{b_n x^n}{\lfloor n} + \dots,$$

then

$$2^r f_r = \frac{x^r}{\lfloor r} + \frac{b_1 x^{r+1}}{\lfloor r+1} + \dots + \frac{b_n x^{n+r}}{\lfloor n+r} + \dots,$$

provided that the successive constants introduced be all zero.

Write $2^r f_r = u_r x^r$, so that

$$u_r = \left(\frac{1}{\lfloor r} + \frac{b_1 x}{\lfloor r+1} + \dots\right);$$

then

$$\sqrt{\mathrm{I}} = u_1 \mathrm{J}_0(z) + \frac{u_2}{2} z\mathrm{J}_1(z) + \frac{u_3}{2^2} z^2 \mathrm{J}_2(z) + \dots, \quad (2)$$

giving the light intensity. Now $b_r = \left(\frac{d^r f}{dx^r}\right)_{x=0}$ so that the b-functions are given by the difference equation

$$b_{r+1} - i\mu b_r - 2i\nu r b_{r-1} = 0.$$

If we write $b_r x^r = \mathrm{A}_r$ then the A functions satisfy the equation

$$\mathrm{A}_{r+1} - (2i\pi n)\,\mathrm{A}_r - (4i\pi n_1)\, r\mathrm{A}_{r-1} = 0,$$

and this is an equation of the form § 3 (2); its solution, therefore, may be written down at once. Moreover the A-functions are independent of the co-ordinate z, and therefore so also are the μ functions; the co-ordinate z occurs only explicitly as shown in (2). If z be small a few terms only of series (2) are required to give the light intensity to a high order of accuracy.

§ 5. Let $n_1 = 0$ in § 3; then $u_2 = 2\pi n$, $\beta_2 = 0$; so that

$$c_{2r} = (-1)^r (2\pi n)^{2r}, \qquad c_{2r+1} = 0, \qquad d_{2r} = 0, \qquad d_{2r+1} = (2\pi n)^{2r+1}(-1)^r;$$

whence

$$2\pi n \mathrm{U} = \frac{4\pi n}{z}\mathrm{J}_1(z) - \left(\frac{4\pi n}{z}\right)^3 \mathrm{J}_3(z) + \ldots,$$

$$2\pi n \mathrm{V} = -\left(\frac{4\pi n}{z}\right)^2 \mathrm{J}_2(z) + \left(\frac{4\pi n}{z}\right)^4 \mathrm{J}_4(z) - \ldots,$$

and these are LOMMEL Functions.*

Again let $n_1 = 0$ in § 4. In this case a choice other than zero for the constants in § 4—in fact, an introduction at each integration of the quantity $\frac{1}{i\mu}$ —will lead to the usual LOMMEL Functions of the second type. The result obtained is :—

$$\mathrm{I} = \left(\frac{1}{4\pi n}\right)^2 \left(\mathrm{V}_0 \cos 2\pi n + \mathrm{V}_1 \sin 2\pi n - \cos\frac{z^2}{8\pi n}\right)^2 + \left(\frac{1}{4\pi n}\right)^2 \left(\mathrm{V}_0 \sin 2\pi n - \mathrm{V}_1 \cos 2\pi n + \sin\frac{z^2}{8\pi n}\right)^2,$$

where

$$\mathrm{V}_r = \left(\frac{z}{4\pi n}\right)^r \mathrm{J}_r(z) - \left(\frac{z}{4\pi n}\right)^{r+2} \mathrm{J}_{r+2}(z) + \ldots + (-1)^{r+m}\left(\frac{z}{4\pi n}\right)^{r+2m} \mathrm{J}_{r+2m}(z) + \ldots.$$

§ 6. Let all the aberration coefficients vanish except σ_2; then the expression $\kappa(\mathrm{U}_1 + r_0)$ takes the form

$$\frac{\kappa\sigma_2}{2}\frac{\rho^3 \mathrm{Y}_1}{d^4}\cos\phi + \frac{\kappa \mathrm{X}}{2d^2}(\rho^2 - 2\mathrm{Y}_1\rho\cos\phi) - \frac{\kappa\rho\rho'}{d}\cos(\phi - \phi'),$$

i.e.,

$$\frac{\kappa \mathrm{X}\rho^2}{2d^2} + \mathrm{C}\cos(\phi - \psi) \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\mathrm{C}\cos\psi = \mathrm{A} - \frac{\kappa\rho\rho'}{d}\cos\phi', \quad \mathrm{C}\sin\psi = -\frac{\kappa\rho\rho'}{d}\sin\phi',$$

and

$$\mathrm{A} = \frac{\kappa\rho}{d}\left(\frac{\sigma_2}{2}\frac{\rho^2\mathrm{Y}_1}{d^3} - \frac{\mathrm{XY}_1}{d}\right), \quad \mathrm{C}^2 = \mathrm{A}^2 - 2\mathrm{A}\frac{\kappa\rho\rho'}{d}\cos\phi' + \left(\frac{\kappa\rho\rho'}{d}\right)^2 \quad . \quad . \quad . \quad (2)$$

Now

$$\int e^{i\kappa(\mathrm{U}_1 + r_0)}\,d\mathrm{S} = \int_0^{\rho_1} \rho e^{\frac{i\kappa \mathrm{X}\rho^2}{2d^2}} d\rho \int_0^{2\pi} e^{i\mathrm{C}\cos(\phi-\psi)}\,d\phi = 2\pi\int_0^{\rho_1} \rho e^{\frac{i\kappa\mathrm{X}\rho^2}{2d^2}} \mathrm{J}_0(\mathrm{C})\,d\rho,$$

* E. Lommel, 'Abh. d. k. Bayer. Akad.,' vol. XV (1886).

ρ_1 being the radius of the exit pupil. Let now

$$z\sqrt{v} = \frac{\kappa\rho\rho'}{d}, \quad z = \frac{\kappa\rho_1\rho'}{d}, \quad \beta = \frac{\kappa\sigma_2}{2}\left(\frac{\rho_1}{d}\right)^3\left(\frac{Y_1}{d}\right), \quad \varepsilon = \kappa X\left(\frac{\rho_1}{d}\right)\left(\frac{Y_1}{d}\right) \text{ and } \mu = \frac{\kappa X}{2}\left(\frac{\rho_1}{d}\right)^2;$$

then substituting in § 1 (3) and using the above results we have I given by the squared modulus of

$$\int_0^1 e^{i\mu v} J_0\left(\sqrt{(\beta v - \varepsilon)^2 v - 2zv(\beta v - \varepsilon)\cos\phi' + z^2 v}\right) dv. \quad . \quad . \quad . \quad . \quad (3)$$

Let now X = 0 and therefore $\mu = 0$, $\varepsilon = 0$; then

$$\sqrt{I} = \int_0^1 J_0(u)\,dv, \quad \text{where} \quad u^2 = z^2 v - 2z\beta v^2\cos\phi' + \beta^2 v^3 = w \text{ (say)}$$

$$= \Big[v J_0(u)\Big]_0^1 + \int_0^1 \frac{J_1(u)}{u}\,d\phi_1, \quad \text{if} \quad d\phi_1 = uv\,du$$

$$= \left[v J_0(u) + \phi_1 \frac{J_1(u)}{u} + \phi_2 \frac{J_2(u)}{u^2} + \ldots\right]_0^1,$$

where

$$d\phi_{n+1} = u\phi_n du, \quad i.e., \quad \frac{2d\phi_{n+1}}{dw} = \phi_n \quad \text{and} \quad \phi_0 = v.$$

Assume now

$$\phi_n = \alpha_{n,1} v + \alpha_{n,2} v^2 + \ldots + \alpha_{n,p} v^p + \ldots$$

so that

$$2(p+1)\,\alpha_{n+1,p+1} = A\alpha_{n,p} + B\alpha_{n,p-1} + C\alpha_{n,p-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\frac{dw}{dv} = A + Bv + Cv^2.$$

Then

$$\sqrt{I} = J_0(u_1)\,\Sigma\alpha_0 + \frac{J_1(u_1)}{u_1}\Sigma\alpha_1 + \frac{J_2(u_1)}{u_1^{\,2}}\Sigma\alpha_2 + \ldots + \frac{J_n(u_1)}{u_1^{\,n}}\Sigma\alpha_n + \ldots \quad . \quad . \quad (5)$$

where

$$u_1^{\,2} = z^2 - 2z\beta\cos\phi' + \beta^2 \quad \text{and} \quad \Sigma\alpha_n = \Sigma_m \alpha_{n,m}; \quad \text{thus } u_1 = PQ. \quad \text{(Part II, § 13.)}$$

This expression gives a value for the intensity and it is especially useful in the regions round $u_1 = 0$ and $u_1 = z\beta$; for here but few terms of the series are required to give the intensity to a high order of accuracy. The quantities $\Sigma\alpha_n$ may be evaluated readily from the difference equation (4); but as only a few are required they are given below :—

$$\Sigma\alpha_0 = 1, \quad \Sigma\alpha_1 = \tfrac{1}{2}\left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\right),$$

$$\Sigma\alpha_2 = \tfrac{1}{2}\left\{\frac{A}{4}\left(\frac{A}{3} + \frac{B}{4} + \frac{C}{5}\right) + \frac{B}{6}\left(\frac{A}{4} + \frac{B}{5} + \frac{C}{6}\right) + \frac{C}{8}\left(\frac{A}{5} + \frac{B}{6} + \frac{C}{7}\right)\right\},$$

and $A = z^2$, $B = -4z\beta\cos\phi'$, $C = 3\beta^2$.

Putting $\beta = 0$
$$\sqrt{\mathrm{I}} = \sum_{n=0}^{\infty} \frac{z^n}{2^n} \frac{\mathrm{J}_n(z)}{\lfloor n+1} = \frac{2\mathrm{J}_1(z)}{z} \qquad \text{(the Airy disc).}$$

§ 7. The above series, although always convergent, is not always suitable for computation. Another series may be obtained as follows :—

$$\sqrt{\mathrm{I}} = \int_0^1 \mathrm{J}_0(u)\,dv = \int_0^1 \frac{1}{u}\frac{dv}{du} d\{u\mathrm{J}_1(u)\}$$
$$= \left[\frac{dv}{du}\mathrm{J}_1(u)\right]_0^1 - \int_0^1 \frac{1}{u}\frac{d\phi_1}{du} d\{u^2\mathrm{J}_2(u)\}, \qquad \text{if } \phi_1 = \frac{1}{u}\frac{dv}{du},$$
$$= 2\frac{dv}{dw}u_1\mathrm{J}_1(u_1) + \ldots + (-1)^{n-1}2^n\frac{d^nv}{dw^n}u_1^n\mathrm{J}_n(u_1) + \ldots, \quad . \quad . \quad (1)$$

since $\phi_{n+1} = \frac{1}{u}\frac{d\phi_n}{du} = 2\frac{d\phi_n}{dw}$; and the series will vanish at the lower limit of integration. This series is useful for the outer parts of the field, owing to its rapid convergence there. The expression $\frac{d^nv}{dw^n}$ may be obtained by direct differentiation of

$$w = z^2v - 2z\beta\, v^2 \cos\phi' + \beta^2 v^3;$$

but again a few terms only are required and they are given below :—

$$\frac{dv}{dw} = \frac{1}{\mathrm{A}+\mathrm{B}+\mathrm{C}}, \quad \frac{d^2v}{dw^2} = -\frac{\mathrm{B}+2\mathrm{C}}{(\mathrm{A}+\mathrm{B}+\mathrm{C})^3}, \quad \frac{d^3v}{dw^3} = \frac{3(\mathrm{B}+2\mathrm{C})^2}{(\mathrm{A}+\mathrm{B}+\mathrm{C})^5} - \frac{2\mathrm{C}}{(\mathrm{A}+\mathrm{B}+\mathrm{C})^4}.$$

Putting $\mathrm{B} = 0$, then $\frac{d^nv}{dw^n} = 0$ if $n>1$; thus $\sqrt{\mathrm{I}} = \frac{2\mathrm{J}_1(z)}{z}$ (the Airy disc).

§ 8. Let all the aberration coefficients vanish except σ_3, σ_4. Then the integration terms in § 1 (1) are given by

$$\mathrm{U}_1 + r_0 = -\frac{\sigma_3}{4d^4}\rho^2\mathrm{Y}_1^2 - \frac{\sigma_4}{2d^4}\rho^2\mathrm{Y}_1^2\cos^2\phi + \frac{\mathrm{X}}{2d^2}(\rho^2 - 2\mathrm{Y}_1\rho\cos\phi) - \frac{\rho\rho'}{d}\cos(\phi-\phi'). \quad (1)$$

In the first place let us write $\sigma_4 = 0$. Then the terms in (1) become

$$\frac{\rho^2}{2d^2}\left(\mathrm{X} - \frac{\sigma_3}{2}\frac{\mathrm{Y}_1^2}{d^2}\right) - \frac{\rho\mathrm{C}}{d}\cos(\phi-\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\left.\begin{aligned} \mathrm{C}\cos\psi &= \mathrm{X}\frac{\mathrm{Y}_1}{d} + \rho'\cos\phi' \\ \mathrm{C}\sin\psi &= \rho'\sin\phi' \end{aligned}\right\}$$

so that

$$\mathrm{C}^2 = \left(\frac{\mathrm{XY}_1}{d}\right)^2 + 2\rho'\left(\frac{\mathrm{XY}_1}{d}\right)\cos\phi' + \rho'^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The integral § 1 (3) becomes

$$\int e^{i\kappa(\mathrm{U}_1+r_0)}\,d\mathrm{S} = \int_0^{\rho_1} \rho\,d\rho \int_0^{2\pi} e^{\frac{i\kappa\rho^2}{2d^2}\left(\mathrm{X}-\frac{\sigma_3}{2}\frac{\mathrm{Y}_1^2}{d^2}\right)-\frac{i\kappa\rho\mathrm{C}}{d}\cos(\phi-\psi)}\,d\phi$$

$$= 2\pi \int_0^{\rho_1} e^{\frac{i\kappa\rho^2}{2d^2}\left(\mathrm{X}-\frac{\sigma_3}{2}\frac{\mathrm{Y}_1^2}{d^2}\right)} \mathrm{J}_0\left(\frac{\kappa\rho\mathrm{C}}{d}\right)\rho\,d\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This integral is of the same form as the integral obtained in studying out-of-focus effects in the absence of spherical aberration; it is the integral which gives rise to LOMMEL functions* and the results already obtained may, therefore, be quoted.

§9. If $\sigma_4 \neq 0$ the expression § 8 (1) may be written in either of the forms

$$\frac{\rho^2}{2d^2}\left(\mathrm{X}-\frac{\sigma_3}{2}\frac{\mathrm{Y}_1^2}{d^2}\right)-\frac{\sigma_4}{2d^4}\rho^2\mathrm{Y}_1^2\cos^2\phi-\frac{\rho\mathrm{C}}{d}\cos(\phi-\psi), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{\rho^2}{2d^2}\left(\mathrm{X}-\frac{\mathrm{Y}_1^2}{2d^2}\overline{\sigma_3+2\sigma_4}\right)+\frac{\sigma_4}{2d^4}\rho^2\mathrm{Y}_1^2\sin^2\phi-\frac{\rho\mathrm{C}}{d}\cos(\phi-\psi). \quad . \quad . \quad . \quad (2)$$

Taking (2) the integral to be considered is of the form

$$\int_0^{\rho_1} e^{i\gamma\rho^2}\rho\,d\rho \int_0^{2\pi} e^{-i\zeta\cos(\phi-\psi)+i\beta\sin^2\phi}\,d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$\gamma\rho^2 = \frac{\kappa\rho^2}{2d^2}\left\{\mathrm{X}-\frac{\mathrm{Y}_1^2}{2d^2}(\sigma_3+2\sigma_4)\right\}, \qquad \zeta = \frac{\kappa\rho\mathrm{C}}{d}, \qquad \beta = \frac{\kappa\rho^2\mathrm{Y}_1^2\sigma_4}{2d^4};$$

while (1) leads to an integral of the form

$$\int_0^{\rho_1} e^{i\gamma\rho^2}\rho\,d\rho \int_0^{2\pi} e^{-i\zeta\cos(\phi-\psi)+i\beta'\cos^2\phi}\,d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where now

$$\gamma'\rho^2 = \frac{\kappa\rho^2}{2d^2}\left\{\mathrm{X}-\frac{\sigma_3}{2}\frac{\mathrm{Y}_1^2}{d^2}\right\}, \qquad \zeta = \frac{\kappa\rho\mathrm{C}}{d}, \qquad \beta' = -\frac{\kappa\rho^2\mathrm{Y}_1^2\sigma_4}{2d^4}.$$

Either of these may be taken as the standard integral for the general case involving σ_3, σ_4.

§ 10. Take § 9 (3) and consider first the integration with respect to ϕ. This may be written

$$\int_0^{2\pi} e^{-i\zeta\cos\phi\cos\psi-i\zeta\sin\phi\sin\psi+i\beta\sin^2\phi}\,d\phi.$$

Now from § 3, the function $e^{ax+\frac{bx^2}{2}}$ may be expanded in powers of x, and the result is

$$e^{ax+\frac{bx^2}{2}} = y_0 + \frac{x}{\underline{|1}}y_1 + \frac{x^2}{\underline{|2}}y_2 + \ldots + \frac{x^n}{\underline{|n}}y_n + \ldots,$$

where y_{n+1} is given by the difference equation

$$y_{n+1} = ay_n + bny_{n-1},$$

* E. Lommel: 'Abh. d. k. Bayer. Akad.,' vol. XV., 1886.

and $y_0 = 1$. The solution of this equation is

$$y_n = a^n + \frac{n^{(2)}}{2} a^{n-2} b + \ldots + n^{(2r)} \frac{a^{n-2r} b^r}{2\,.\,4\,\ldots\,2r} + \ldots. \qquad (1)\ \S\, 3.$$

If, therefore, we write

$$x = \sin\phi, \quad a = -i\zeta\sin\psi, \quad b = 2i\beta, \quad \text{then}$$

$$\int_0^{2\pi} e^{-i\zeta\cos(\phi-\psi)+i\beta\sin^2\phi}\,d\phi = \sum_{n=0}^{\infty} \frac{y_n}{\lfloor n} \int_0^{2\pi} e^{-i\zeta\cos\phi\cos\psi} \sin^n\phi\, d\phi$$

$$= \sum_{n=0}^{\infty} \frac{y_{2n}}{\lfloor 2n} \int_0^{2\pi} e^{-i\zeta\cos\phi\cos\psi} \sin^{2n}\phi\, d\phi$$

$$^* = 2\pi \sum_{n=0}^{\infty} \frac{y_{2n}}{2^n \lfloor n} \frac{J_n(\zeta\cos\psi)}{(\zeta\cos\psi)^n}, \quad \ldots \ldots \quad (2)$$

since the integrals vanish which contain an odd power of $\sin\phi$. Now β contains ρ^2, a and ζ each contain ρ; therefore, y_{2n} is of the form $\rho^{2n} Y_{2n}$ where Y_{2n} does not contain ρ. Write $\zeta \cos \psi = Z$, then

$$y_{2n} = \left(\frac{d}{\kappa C \cos\psi}\right)^{2n} Z^{2n} Y_{2n} = Z^{2n} Y'_{2n} \quad (\text{say})$$

and (2) becomes

$$2\pi \sum_0^{\infty} \frac{1}{2^n \lfloor n} Y'_{2n} Z^n J_n(Z). \quad \ldots \ldots \quad (3)$$

The general integral is, therefore,

$$2\pi \int_0^{\rho_1} \left\{ \sum_{n=0}^{\infty} \frac{1}{2^n \lfloor n} Y'_{2n} Z^n J_n(Z) \right\} \rho e^{i\gamma\rho^2} d\rho = 2\pi \left(\frac{\rho_1}{Z_1}\right)^2 \sum_{n=0}^{\infty} \frac{1}{2^n \lfloor n} Y'_{2n} \int_0^{Z_1} Z^{n+1} e^{i\nu Z^2} J_n(Z)\, dZ, \quad (4)$$

where $\nu Z^2 = \gamma\rho^2$ and the variable of integration has been changed from ρ to Z. Omitting now the factor $\pi\rho_1^2$, so that the intensity shall be unity at the Gaussian point when $\sigma_3 = \sigma_4 = 0$, the result is obtained that the general intensity is given by the squared modulus of the expression

$$\frac{2}{Z^2} \sum_{n=0}^{\infty} \frac{1}{2^n \lfloor n} Y'_{2n} U_{n+1}, \quad \ldots \ldots \quad (5)$$

where

$$U_{n+1} = \int_0^{Z_1} Z^{n+1} e^{i\nu Z^2} J_n(Z)\, dZ,$$

and is an ordinary LOMMEL function.

The above result has been obtained from § 9 (3). An equivalent result and of identically the same form can be derived from § 9 (4), but the symbols will have slightly different meaning, *e.g.*, γ', β' will occur instead of γ, β and $\zeta \sin \psi$ instead of $\zeta \cos \psi$. The result will follow from the BESSEL function formula

$$J_n(z) = \left(\frac{z}{2}\right)^n \frac{1}{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-\pi/2}^{\pi/2} e^{\pm iz\sin\phi} \cos^{2n}\phi\, d\phi,$$

valid if $R(n+\frac{1}{2}) > 0$.

* Assuming the formula $J_n(z) = \dfrac{z^n}{2^n\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^{\pi} e^{\pm iz\cos\phi} \sin^{2n}\phi\, d\phi$, valid if $R(n+\frac{1}{2}) > 0$.

§11. Let all the aberration coefficients vanish except σ_5. Then

$$U_1 + r_0 = \frac{\sigma_5}{2d^4}\rho Y_1^3 \cos\phi + \frac{X}{2d^2}(\rho^2 - 2Y_1\rho\cos\phi) - \frac{\rho\rho'}{d}\cos(\phi - \phi').$$

In the first place let $X = 0$. Then

$$U_1 + r_0 = \frac{\rho C}{d}\cos(\phi - \psi)$$

where

$$\left.\begin{aligned} C\cos\psi &= \frac{\sigma_5}{2d^3}Y_1^3 - \rho'\cos\phi' \\ C\sin\psi &= \qquad\quad -\rho'\sin\phi' \end{aligned}\right\}.$$

Thus the integral § 1 (3) becomes

$$\int_0^{\rho_1}\rho d\rho \int_0^{2\pi} e^{\frac{i\kappa\rho C}{d}\cos(\phi-\psi)}\, d\phi,$$

where C is constant so far as ρ, ϕ are concerned. This is the AIRY disc integral, so that the light intensity at the point given by (C, ψ) is

$$\left\{\frac{2J_1\left(\frac{\kappa\rho C}{d}\right)}{\left(\frac{\kappa\rho C}{d}\right)}\right\}^2 . \quad (1)$$

Let now $X \neq 0$. Then

$$U_1 + r_0 = \frac{X}{2}\frac{\rho^2}{d^2} + \frac{\rho C'}{d}\cos(\phi - \psi'), \quad (2)$$

where

$$C'^2 = \left(\frac{\sigma_5}{2d^3}Y_1^3 - X\frac{Y_1}{d}\right)^2 - 2\rho'\left(\frac{\sigma_5 Y_1^3}{2d^3} - X\frac{Y_1}{d}\right)\cos\phi + \rho'^2,$$

and ψ' is independent of ρ, ϕ. Thus the general integral of § 1 (3) becomes

$$\int_0^{\rho_1} e^{i\kappa\frac{X}{2}\frac{\rho^2}{d^2}}\rho d\rho \int_0^{2\pi} e^{i\kappa\frac{\rho C'}{d}\cos(\phi-\psi')}\, d\phi,$$

and we are led to

$$\int_0^{\rho_1} e^{\frac{i\kappa\rho^2 X}{2d^2}} J_0\left(\frac{\kappa\rho C'}{d}\right)\rho d\rho, \quad (3)$$

and this is the integral which occurred in the consideration of out-of-focus effects in the absence of aberration. It leads to ordinary LOMMEL functions and the results may, therefore, be quoted. Put $\kappa X\rho^2 = 4\pi n d^2$, $\kappa\rho C' = zd$. Then

$$\frac{2}{z^2}\int_0^z e^{i\mu t^2} J_0(t)\, t\, dt = \frac{e^{2\pi i n}}{2\pi i n}\left\{(4\pi i n)\frac{J_1(z)}{z} - (4\pi i n)^2\frac{J_2(z)}{z^2} + \dots\right\}. \quad (4)$$